Generally useful constants (MKS):

In[8]:=

```
amu = 1.6605 (10^-27) kg
NA = 6.02221 (10^23) / mole
k = 1.38066 (10^-23) J / K
me = 9.1094 (10^-31) kg
h = 6.62608 (10^-34) J s
q = 1.6022 (10^-19) C
c = 2.9979 (10^8) m / s
```

Out[8]= $1.6605 \times 10^{-27}\ \mathrm{kg}$

Out[9]= $\dfrac{6.02221 \times 10^{23}}{\mathrm{mole}}$

Out[10]= $\dfrac{1.38066 \times 10^{-23}\ \mathrm{J}}{\mathrm{K}}$

Out[11]= $9.1094 \times 10^{-31}\ \mathrm{kg}$

Out[12]= $6.62608 \times 10^{-34}\ \mathrm{J\ s}$

Out[13]= $1.6022 \times 10^{-19}\ \mathrm{C}$

Out[14]= $\dfrac{2.9979 \times 10^{8}\ \mathrm{m}}{\mathrm{s}}$

Problem 1.12 (Engel)

Part A

Show that the energy density radiated by a blackbody depends on the temperature as T^4. The starti
that gives energy density as a function of frequency (ν) and temperature (T).

$$\rho = \int_0^\infty \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \, d\nu$$

Solution

Strategy. The problem contains a hint: transform the integral by substituting a new variable, x, for
$= \frac{h\nu}{kT}$. Then use a definite integral formula provided in the hint to simplify the result.

To make a substitution of variables, we must do *three* things:
1) express ν as a function of x and replace ν everywhere in the integral with this function
2) express dν as a function of dx and make this substitution
3) express the limits of integration ($\nu = 0 \rightarrow \nu = \infty$) in terms of the corresponding values of x

Execution. Substitution of variables requires the 3 steps listed above. First, however, we must def
useful expressions:

$$\nu = \frac{kTx}{h}$$

$$\frac{d\nu}{dx} = \frac{kT}{h}$$

$$d\nu = \frac{kT}{h} \, dx$$

Step #1. First, we transform the formula inside the integral by replacing ν with $\frac{kTx}{h}$. This gives:

$$\frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$\frac{8\pi h (kTx/h)^3}{c^3} \frac{1}{e^x - 1}$$

$$\frac{8\pi (kT)^3}{h^2 c^3} \frac{x^3}{e^x - 1}$$

Step #2. Next, we replace the index of integration, dν, with the corresponding expression in dx. Th
but here is the result again:

$$d\nu = \frac{k\,T}{h}\,dx$$

Step #3. The integration with respect to ν runs from $\nu = 0$ to $\nu = \infty$. We need to replace these values
0 to $\nu = \infty$. As it happens, $x = 0$ makes $\nu = 0$, and $x = \infty$ makes $\nu = \infty$. In other words, we need to inte

Putting it all together:

$$\int_0^\infty \frac{8\,\pi\,h\,\nu^3}{c^3}\;\frac{1}{e^{\frac{h\,\nu}{k\,T}}-1}\,d\nu$$

$$\int_0^\infty \frac{8\,\pi\,(k\,T)^3}{h^2\,c^3}\;\frac{x^3}{e^x-1}\;\frac{k\,T}{h}\,dx$$

$$\int_0^\infty \frac{8\,\pi\,(k\,T)^4}{h^3\,c^3}\;\frac{x^3}{e^x-1}\,dx$$

$$\frac{8\,\pi\,(k\,T)^4}{h^3\,c^3}\int_0^\infty \frac{x^3}{e^x-1}\,dx$$

$$\frac{8\,\pi\,(k\,T)^4}{h^3\,c^3}\;\frac{\pi^4}{15}$$

$$\frac{8\,\pi^5\,(k\,T)^4}{15\,(h\,c)^3}$$

We ultimately arrive (4th line) at a formula that contains a definite integral that can be evaluated usi
the last two lines which no longer contain an integral, and these show the T^4dependence of the energ

Note, it is also possible to evaluate the definite integral using *Mathematica* using two different metho

In[15]:=

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx$$

$$\text{NIntegrate}\left[\frac{x^3}{e^x-1},\ \{x,\ 0,\ \infty\}\right]$$

Out[15]=

$$\frac{\pi^4}{15}$$

Out[16]=

```
6.49394
```

Part B

Use the new function obtained in part A to calculate the energy density radiated by a blackbody at 80

Solution

Strategy. Just evaluate this formula at the two temperatures:

$$\frac{8\,\pi^5\,(k\,T)^4}{15\,(h\,c)^3}$$

Execution.

In[17]:=

$$\frac{8\,\pi^5\,(k\,\{800\,K,\ 4000\,K\})^4}{15\,(h\,c)^3}$$

Out[17]=

$$\left\{\frac{0.000309909\,J}{m^3},\ \frac{0.193693\,J}{m^3}\right\}$$

Comment

Ideal blackbodies sound like an esoteric object that is sure to be unimportant to anyone other tha conclusion. We actually run into things that approach blackbody behavior all the time.

All matter radiates energy according to the T^4 law seen here. You, me, and the deep blue sea. Cons object in the lab, a distillation in the organic lab, the solid state furnace in the inorganic lab, the obje T^4. <u>High temperature processes waste a lot of energy!</u>

You may have seen glassware, like distillation columns, that are coated with a thin layer of silver pretty). The idea here is to *reflect* the radiated energy back into the apparatus. The rationale for mathematics seen in this problem.