Problem 2.11 (Engel)

Determine in each case whether the function is an eigenfunction of the operator. If it is, what is the eigenvalue?

A. function = x^3 and operator = d^3 / dx^3

Solution.

$\frac{d^3}{dx^3} x^3 =$

$= \frac{d}{dx} \left[\frac{d}{dx} \left[\frac{d}{dx} x^3 \right]\right]$

$= \frac{d}{dx} \left[\frac{d}{dx} 3x^2 \right]$

$= \frac{d}{dx} 6x$

= 6

This result is not a scalar multiple of x^3 so the function is not an eigenfunction of this operator.

B. function = $x\,y$ and operator = $x(\partial / \partial x) + y(\partial / \partial y)$

Solution.

$\frac{\partial}{\partial x} x\,y = y$

$\frac{\partial}{\partial y} x\,y = x$

$\left[x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right] x\,y = x\,(y) + y\,(x) = 2\,x\,y$

This result is a scalar multiple of $x\,y$ so the function is an eigenfunction. The eigenvalue is 2.

C. function = $\sin\theta \cos\phi$ and operator = $\partial^2 / \partial\phi^2$

Solution.

$\frac{\partial^2}{\partial\phi^2} \sin\theta \cos\phi = \sin\theta \frac{\partial^2}{\partial\phi^2} \cos\phi$

$= \sin\theta \frac{\partial}{\partial\phi} \frac{\partial}{\partial\phi} \cos\phi$

$= \sin\theta \frac{\partial}{\partial\phi} [-\sin\phi]$

$= \sin\theta [-\cos\phi]$

= -1 ($\sin\theta \cos\phi$)

This result is a scalar multiple of $\sin\theta \cos\phi$ so the function is an eigenfunction. The eigenvalue is -1.