
Problem 2.27 (Engel)

Express the function $f(x) = x$ that is restricted to the interval $x: -b \rightarrow b$ using a five-term Fourier series expansion:

$$f(x) = d_0 + \sum_{n=1}^5 (c_n \text{Sin}\left[\frac{n\pi x}{b}\right] + d_n \text{Cos}\left[\frac{n\pi x}{b}\right])$$

Solution

Strategy.

The values of c_n and d_n are defined by these integrals:

$$c_n = \frac{\int_{-b}^b f(x) \text{Sin}\left[\frac{n\pi x}{b}\right] dx}{\int_{-b}^b \text{Sin}^2\left[\frac{n\pi x}{b}\right] dx}$$

and

$$d_n = \frac{\int_{-b}^b f(x) \text{Cos}\left[\frac{n\pi x}{b}\right] dx}{\int_{-b}^b \text{Cos}^2\left[\frac{n\pi x}{b}\right] dx}$$

$$\mathbf{sines} = \text{Table}\left[\text{Sin}\left[\frac{\mathbf{n} \pi \mathbf{x}}{\mathbf{b}}\right], \{\mathbf{n}, 1, 5\}\right]$$

$$\mathbf{cosines} = \text{Table}\left[\text{Cos}\left[\frac{\mathbf{n} \pi \mathbf{x}}{\mathbf{b}}\right], \{\mathbf{n}, 1, 5\}\right]$$

$$\left\{\text{Sin}\left[\frac{\pi x}{b}\right], \text{Sin}\left[\frac{2\pi x}{b}\right], \text{Sin}\left[\frac{3\pi x}{b}\right], \text{Sin}\left[\frac{4\pi x}{b}\right], \text{Sin}\left[\frac{5\pi x}{b}\right]\right\}$$

$$\left\{\text{Cos}\left[\frac{\pi x}{b}\right], \text{Cos}\left[\frac{2\pi x}{b}\right], \text{Cos}\left[\frac{3\pi x}{b}\right], \text{Cos}\left[\frac{4\pi x}{b}\right], \text{Cos}\left[\frac{5\pi x}{b}\right]\right\}$$

$$\mathbf{c} = \frac{\int_{-b}^b \mathbf{x} \mathbf{sines} \, d\mathbf{x}}{\int_{-b}^b \mathbf{sines}^2 \, d\mathbf{x}}$$

$$\left\{\frac{2b}{\pi}, -\frac{b}{\pi}, \frac{2b}{3\pi}, -\frac{b}{2\pi}, \frac{2b}{5\pi}\right\}$$

$$\mathbf{d} = \frac{\int_{-b}^b \mathbf{x} \mathbf{cosines} \, d\mathbf{x}}{\int_{-b}^b \mathbf{cosines}^2 \, d\mathbf{x}}$$

$$\{0, 0, 0, 0, 0\}$$

The cos functions vanish from the expansion because $d_n = 0$ for $n = 1-5$.

To determine d_0 we notice that this term equals $f(0)$ because all of the sin functions vanish at $x = 0$. $f(0) = 0$, so $d_0 = 0$.

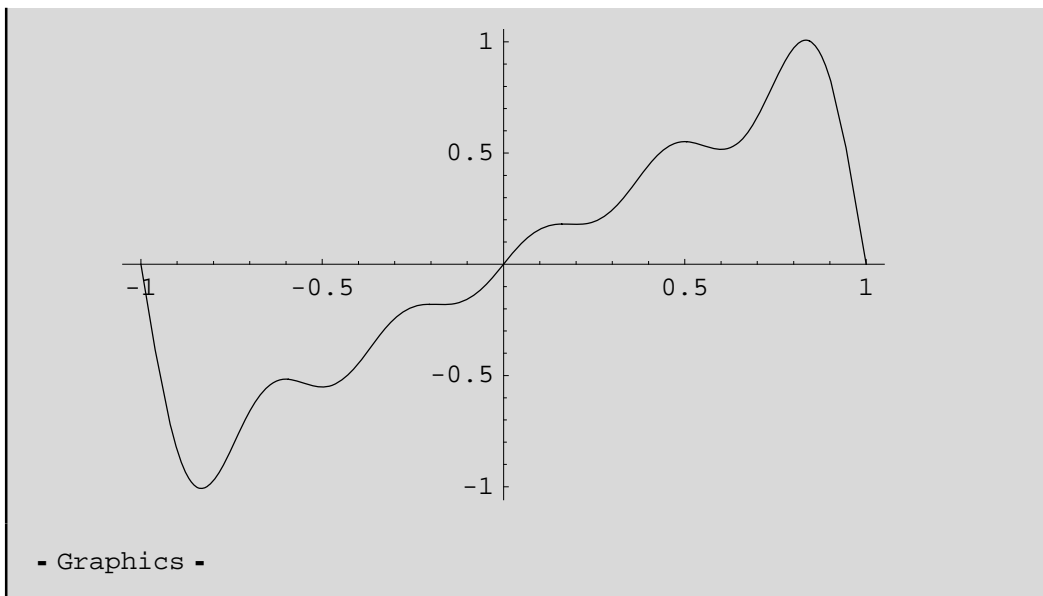
We can recreate the Fourier series and plot it using a dot product to combine the coefficients and the sin functions.

```
f = c.sines
```

$$\frac{2b \sin\left[\frac{\pi x}{b}\right]}{\pi} - \frac{b \sin\left[\frac{2\pi x}{b}\right]}{\pi} + \frac{2b \sin\left[\frac{3\pi x}{b}\right]}{3\pi} - \frac{b \sin\left[\frac{4\pi x}{b}\right]}{2\pi} + \frac{2b \sin\left[\frac{5\pi x}{b}\right]}{5\pi}$$

```
b = 1
Plot[f, {x, -1, 1}]

1
```



Comment. We could save ourselves the "trouble" of calculating the d_n by noticing that $f(x)$ is an odd function over the interval $-b \rightarrow b$, and the cos functions are even functions. The product ($f \cos$) must be odd, and its integral must vanish over the interval.

Observation. The five-term expansion doesn't look much like $f(x) = x$. Let's look at a 10 term expansion:

```

b = 1
sines = Table[Sin[ $\frac{n \pi x}{b}$ ], {n, 1, 10}]

c =  $\frac{\int_{-b}^b x \text{sines} dx}{\int_{-b}^b \text{sines}^2 dx}$ 

Plot[c.sines, {x, -1, 1}]

1

```

```

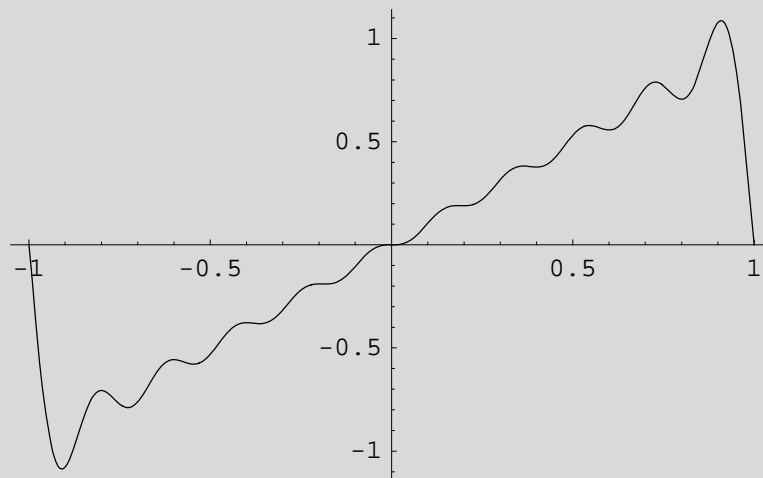
{Sin[ $\pi x$ ], Sin[ $2 \pi x$ ], Sin[ $3 \pi x$ ], Sin[ $4 \pi x$ ], Sin[ $5 \pi x$ ],
Sin[ $6 \pi x$ ], Sin[ $7 \pi x$ ], Sin[ $8 \pi x$ ], Sin[ $9 \pi x$ ], Sin[ $10 \pi x$ ]}

```

```

{ $\frac{2}{\pi}$ ,  $-\frac{1}{\pi}$ ,  $\frac{2}{3\pi}$ ,  $-\frac{1}{2\pi}$ ,  $\frac{2}{5\pi}$ ,  $-\frac{1}{3\pi}$ ,  $\frac{2}{7\pi}$ ,  $-\frac{1}{4\pi}$ ,  $\frac{2}{9\pi}$ ,  $-\frac{1}{5\pi}$ }

```



- Graphics -

And let's look at a 15 term expansion:

```

b = 1
sines = Table[Sin[ $\frac{n \pi x}{b}$ ], {n, 1, 15}]
c =  $\frac{\int_{-b}^b x \text{sines} dx}{\int_{-b}^b \text{sines}^2 dx}$ 
Plot[c.sines, {x, -1, 1}]
1

```

```

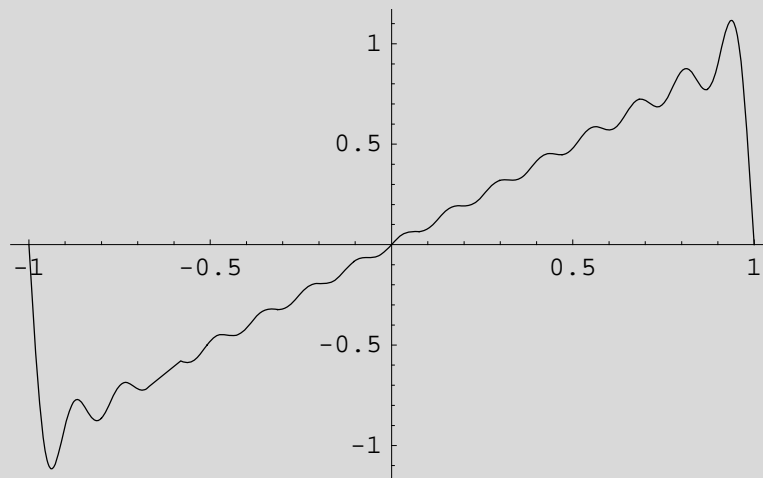
{Sin[ $\pi x$ ], Sin[ $2 \pi x$ ], Sin[ $3 \pi x$ ], Sin[ $4 \pi x$ ], Sin[ $5 \pi x$ ],
Sin[ $6 \pi x$ ], Sin[ $7 \pi x$ ], Sin[ $8 \pi x$ ], Sin[ $9 \pi x$ ], Sin[ $10 \pi x$ ],
Sin[ $11 \pi x$ ], Sin[ $12 \pi x$ ], Sin[ $13 \pi x$ ], Sin[ $14 \pi x$ ], Sin[ $15 \pi x$ ]}

```

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{ $\frac{2}{\pi}$ ,  $-\frac{1}{\pi}$ ,  $\frac{2}{3\pi}$ ,  $-\frac{1}{2\pi}$ ,  $\frac{2}{5\pi}$ ,  $-\frac{1}{3\pi}$ ,  $\frac{2}{7\pi}$ ,
 $-\frac{1}{4\pi}$ ,  $\frac{2}{9\pi}$ ,  $-\frac{1}{5\pi}$ ,  $\frac{2}{11\pi}$ ,  $-\frac{1}{6\pi}$ ,  $\frac{2}{13\pi}$ ,  $-\frac{1}{7\pi}$ ,  $\frac{2}{15\pi}$ }

```



- Graphics -

The quality of the expansion as a fitting function diminishes as the function approaches the edges of the interval. This is *unavoidable* in this case, but not in all cases. The problem is this: $f(x) = x$ does not vanish at the edges, but our Fourier expansion is based on sin functions that must vanish at the edges. It would make better sense to expand $f(x)$ using functions that satisfy the same boundary conditions as $f(x)$.