Problem 2.11 (Lowe)

Prove the following statement: Any linear combination of *degenerate* eigenfunctions of $\hat{H}$ is also an eigenfunction of $\hat{H}$.

Proof

Suppose we have two eigenfunctions of $\hat{H}$ with the same eigenvalue λ. In other words, there exists:

$\hat{H}\psi = \lambda\psi$ and $\hat{H}\phi = \lambda\phi$

Now define a third function that is a linear combination of the first two:

$$\chi = a\psi + b\phi$$

We want to prove:

$$\hat{H}\chi = \lambda\chi$$

Let's start by expanding the left side of this equation:

$$\hat{H}\chi = \hat{H}(a\psi + b\phi)$$

Quantum mechanical operators are *linear*, so

$$\hat{H}\chi = \hat{H}(a\psi + b\phi) = \left(a\hat{H}\psi + b\hat{H}\phi\right)$$

$$= a\lambda\psi + b\lambda\phi = \lambda(a\psi + b\phi) = \lambda\chi$$