Problem 4.10 (Engel)

Part A. Are the eigenfunctions of $\hat{H}$ for the particle in a one-dimensional box also eigenfunctions of the momentum operator $\hat{p}_x$?

Solution

Strategy. The eigenfunctions of $\hat{H}$ are $\psi(x) = \sqrt{\frac{2}{a}} \sin[\frac{n \pi x}{a}]$. If these are also eigenfunctions of $\hat{p}_x$, they will satisfy $\hat{p}_x \psi = \text{const} * \psi$, so let's evaluate the left side of this equation and see what happens.

Execution.

$\hat{p}_x = -i \,\text{hbar} \frac{\partial}{\partial x}$

$\hat{p}_x \psi = -i \,\text{hbar} \frac{\partial \psi}{\partial x}$

The derivative can be evaluated as follows:

$$\partial_x \left(\sqrt{\frac{2}{a}} \, \text{Sin}\left[\frac{n \pi x}{a}\right] \right)$$

$$\sqrt{2} \left(\frac{1}{a}\right)^{3/2} n \pi \, \text{Cos}\left[\frac{n \pi x}{a}\right]$$

The result is a cosine function, which is obviously not a constant multiple of the original sine function. ψ is not an eigenfunction of $\hat{p}_x$.

Part B. Calculate the average value of p_x for the case where n = 3 and 5, i.e., $\psi(x) = \sqrt{\frac{2}{a}} \sin[\frac{3 \pi x}{a}]$ or $\sqrt{\frac{2}{a}} \sin[\frac{5 \pi x}{a}]$

Solution

Strategy. The average value of p_x is also called the expectation value, $<p_x>$, and is obtained as the value of the following integral:

$$<x> = \int_0^a \psi \, \hat{p}_x \, \psi \, dx$$

Note: this integral relies on these facts: 1) ψ is a real function, and 2) ψ has been normalized.

Execution.

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n = {3, 5}
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$$\psi = \sqrt{\frac{2}{a}}\,\mathrm{Sin}\left[\frac{n\,\pi\,x}{a}\right]$$

{3, 5}

$$\left\{\sqrt{2}\,\sqrt{\frac{1}{a}}\,\mathrm{Sin}\left[\frac{3\,\pi\,x}{a}\right],\ \sqrt{2}\,\sqrt{\frac{1}{a}}\,\mathrm{Sin}\left[\frac{5\,\pi\,x}{a}\right]\right\}$$

$$p\psi = (-i\,\mathrm{hbar})\,\partial_x\,\psi$$

$$\left\{-3\,i\,\sqrt{2}\,\left(\frac{1}{a}\right)^{3/2}\mathrm{hbar}\,\pi\,\mathrm{Cos}\left[\frac{3\,\pi\,x}{a}\right],\ -5\,i\,\sqrt{2}\,\left(\frac{1}{a}\right)^{3/2}\mathrm{hbar}\,\pi\,\mathrm{Cos}\left[\frac{5\,\pi\,x}{a}\right]\right\}$$

$$\psi p\psi = \psi * p\psi$$

$$\left\{-\frac{6\,i\,\mathrm{hbar}\,\pi\,\mathrm{Cos}\left[\frac{3\pi x}{a}\right]\mathrm{Sin}\left[\frac{3\pi x}{a}\right]}{a^2},\ -\frac{10\,i\,\mathrm{hbar}\,\pi\,\mathrm{Cos}\left[\frac{5\pi x}{a}\right]\mathrm{Sin}\left[\frac{5\pi x}{a}\right]}{a^2}\right\}$$

$$\int_0^a \psi p\psi \, dx$$

{0, 0}

The average momentum of the particle for both the $n = 3$ and $n = 5$ states is zero. You may have noticed that both of the expressions that needed to be integrated contained cos*sin and was an odd function, so the integrals were guaranteed to vanish.

Comment. The text would like us to generalize this result to the following: the average momentum of the particle in a box is zero regardless of n. You might be reluctant to make this generalization, however, because the two examples that were chosen both involved odd values of n. To guarantee the result, we calculate the average value for an arbitrary case of $n = m$

$$\psi = \sqrt{\frac{2}{a}}\ \mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]$$

$$\sqrt{2}\ \sqrt{\frac{1}{a}}\ \mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]$$

$$p\psi = (-i\ \mathrm{hbar})\ \partial_x\ \psi$$

$$-i\ \sqrt{2}\ \left(\frac{1}{a}\right)^{3/2}\ \mathrm{hbar}\ m\ \pi\ \mathrm{Cos}\left[\frac{m\,\pi\,x}{a}\right]$$

$$\psi p\psi = \psi * p\psi$$

$$-\frac{2\ i\ \mathrm{hbar}\ m\ \pi\ \mathrm{Cos}\left[\frac{m\,\pi\,x}{a}\right]\ \mathrm{Sin}\left[\frac{m\,\pi\,x}{a}\right]}{a^2}$$

$$\int_0^a \psi p\psi\ dx$$

$$-\frac{i\ \mathrm{hbar}\ \mathrm{Sin}[m\,\pi]^2}{a}$$

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Simplify[%, m ∈ Integers]
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