Problem 4.7 (Engel)

Use the eigenfunction $\psi(x) = A' e^{+ikx} + B' e^{-ikx}$to apply the boundary conditions for the particle in a box.

Part A. How do the boundary conditions restrict the acceptable choices for A', B' and k?

Solution

Execution.

Both exponentials equal 1 at $x = 0$ but the boundary condition requires $\psi(0) = 0$.

$\psi(0) = 0 = A' + B'$

Which leads to $B' = -A'$, and simplifies $\psi(x)$ as follows:

$\psi(x) = A'(e^{+ikx} - e^{-ikx})$

The second boundary condition $\psi(a) = 0$ leads to:

$\psi(a) = 0 = A'(e^{+ika} - e^{-ika})$

$e^{+ika} = e^{-ika}$

This says that k must be chosen so that e^{+ika} is real (then it will equal its complex conjugate). In other words, $\sin(k\,a) = 0$. This will happen if

$k\,a = n\,\pi$ where n = integer

or

$k = \frac{n\pi}{a}$

$\psi(x) = A'(e^{+in\pi x/a} - e^{-in\pi x/a})$

Part B.

Do this eigenfunction, and the traditional $A\sin(n\,\pi\,x/a)$, give different probability densities when each function is normalized?

Solution

Execution.

First we determine the values of the normalization constants, A and A'.

P4.6 showed us that $A = \sqrt{\frac{2}{a}}$. The probability density for this eigenfunction is:

$$= [A \sin(\frac{n \pi x}{a})]^2 = \frac{2}{a} [\sin(\frac{n \pi x}{a})]^2$$

Now working on A'

$$\psi(x) = A' (e^{+ i n \pi x/a} - e^{- i n \pi x/a})$$
$$\psi^*(x) = A' (e^{- i n \pi x/a} - e^{+ i n \pi x/a})$$

$$\psi^*(x) \psi(x) = (A')^2 (e^{+ i n \pi x/a} - e^{- i n \pi x/a}) (e^{- i n \pi x/a} - e^{+ i n \pi x/a})$$

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FullSimplify[(e^(i n π x/a) - e^(-i n π x/a)) (e^(-i n π x/a) - e^(i n π x/a))]
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$$4 \mathrm{Sin}[\frac{n \pi x}{a}]^2$$

$$\psi^*(x) \psi(x) = (A')^2 4 [\mathrm{Sin}(\frac{n \pi x}{a})]^2$$

$$4 \int_0^a \mathrm{Sin}[\frac{n \pi x}{a}]^2 dx$$

$$a \left(2 - \frac{\mathrm{Sin}[2 n \pi]}{n \pi}\right)$$

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Simplify[%, n ∈ Integers]
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2 a
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The normalization integral equals one, so $A' = \frac{1}{\sqrt{2 a}}$

The probability density becomes:

$$\psi^*(x) \psi(x) = \left(\frac{1}{\sqrt{2 a}}\right)^2 4 [\mathrm{Sin}(\frac{n \pi x}{a})]^2 = \frac{2}{a} [\mathrm{Sin}(\frac{n \pi x}{a})]^2$$

This result is identical to the previous one, so both ways of writing the eigenfunction are equivalent.