Laboratory I: Impedance

Reading: Lab1_handout
              Simpson   Chapter 1 (if necessary) & Chapter 2 (particularly 2.9-2.13)

1   Lab Exercises

1.1   Part I

What is the input impedance of the oscilloscope on your lab bench? The scope's input behaves as a resistor R in parallel with a capacitor C. Its internal circuitry measures the voltage across this parallel combination. Determine R and C by some appropriate method.
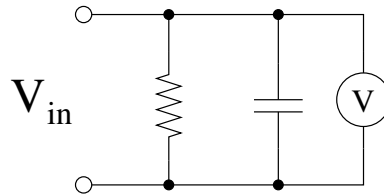

Figure 1: Model for an oscilloscope input.

1.2   Part II

For *at least* four different black boxes (out of A,B,C,D,E) determine:

1. What is inside each black box?

2. What is the arrangement of the circuit inside (how are things hooked up)?

3. Quantitatively, what are the values of the components?

In your lab write-up: Be sure to tell me the reasoning behind your answers and your experimental technique. It is not enough to just state your answer. Also, make sure you record and report to me which black box you used (they are all slightly different).

(A) A single element.

(B) Two elements.

(C) Two elements.

(D) One element.

(E) A small resistance (a copper wire). Try to avoid measuring the resistance of your leads.

2   Some hints & examples

There are two parts to this lab. First, you are asked to determine the input impedances of your lab oscilloscope. In doing this, it is my hope that you will learn (or relearn) a lesson that many people only find out the hard way: lab instruments all too often interact with the system you are working with. If you cannot say, "I know that my instrument is having a negligible affect on my system, then at least you should be able to say, "I know exactly how my instrument is affecting my system.

In the second part of this lab you are asked to try determining the contents of a few "black boxes." The boxes contain some circuit elements – resistors, capacitors, inductors – and it is up to you to figure out which circuit elements are in there, how they are connected, and the values of the resistor(s), capacitor(s), inductor(s) that are inside.

I recommend that you always start by using an ohmmeter to find out how the circuit behaves at DC, that is, determine $|Z_{eff}(\omega \to 0)|$. Then put the box in series or parallel with another known component and a signal generator. If you put the box in series with a resistor (or capacitor, or inductor), for example, you will have a relatively simple voltage divider. What happens as you systematically increase the signal frequency by factors of 10? What is $|Z_{eff}(\omega \to \infty)|$?

It might be helpful to change the value of the known component and see how the voltage across the box (or the known component) changes. It can also be helpful to put both the signal into the voltage divider and the signal out of the voltage divider on the scope at the same time. Measuring the phase shift can be tricky (make sure you are triggering the scope properly) and imprecise, but it can give you help in answering questions (1) and (2). [Note: Phase measurements also can be determined using the X-Y option on the scope. If you are interested, I'll discuss this with you individually.] Once you know the answer to questions (1) and (2), question (3) is best answered by making (at least five) measurements and using the slope and/or intercept from a straight line plot. This may require some clever thinking! Remember also what you learned in Part I of the lab–your measurement apparatus has a finite impedance that may or may not be important.

2.1   Example 1

Here's an example of how to do a black box:

> I am working with a box that I know has only one component.

  I. I use the ohmmeter (HP 34401A) to measure the DC resistance, and I find that the resistance is very low. From this I determine that I have either a very small resistor, or an inductor inside the box (why?).

 II. I place a resistor in series with the box, apply a sinusoidal voltage to the series combination and measure the voltage across the resistor.

> Figure 2 shows the detailed experimental setup. A simplified version of the setup is shown in Figure 3. The simplified view is sufficient, if I assume that the output impedance of the function generator, the properties of the coaxial cables (resistance, capacitance), and the input impedance of the oscilloscope do not influence my measurement. Also, I only measure amplitudes and do not attempt to determine the phase shift.

> I notice that as I increase the frequency of the input voltage, the output voltage decreases. Therefore, the box must contain an inductor, not a resistor (why?).
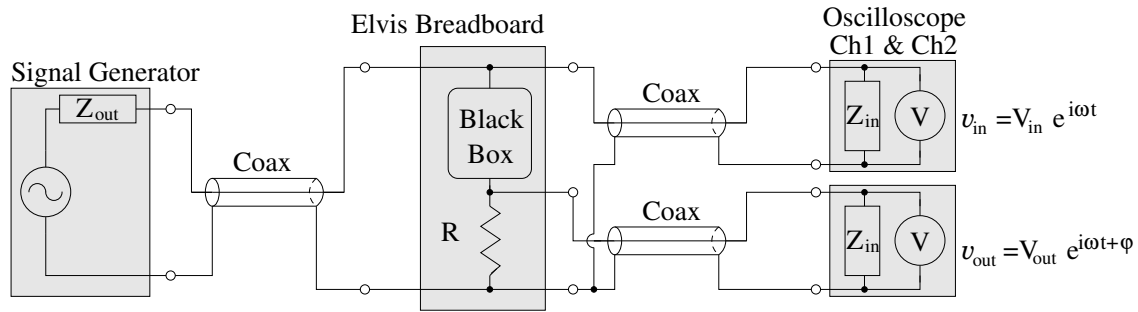
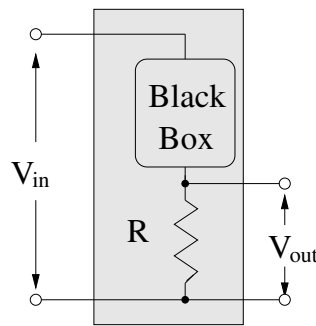Figure 2: Possible setup for measuring a black box. Detailed.

Figure 3: Possible setup for measuring a black box. Same as Fig. 2 but simplified.

III. Since I now know there is an inductor inside the box, I calculate what the output voltage must be:

$$V_{out} = \frac{R}{R + i\omega L} V_{in}$$

$$|V_{out}|^2 = \frac{R^2}{R^2 + \omega^2 L^2} |V_{in}|^2$$

$$R^2 + \omega^2 L^2 = \frac{R^2 |V_{in}|^2}{|V_{out}|^2}$$

$$1 + \left(\frac{\omega^2}{R^2}\right) L^2 = \frac{|V_{in}|^2}{|V_{out}|^2} \tag{1}$$

$$\left(\frac{\omega}{R}\right) L = \sqrt{\frac{|V_{in}|^2}{|V_{out}|^2} - 1} \tag{2}$$

Eq. (1) and Eq. (2) suggest that one can determine L (i) by fixing the frequency and observing how V_{out} varies as a function of resistance R or (ii) by fixing the resistance R and observing how V_{out} varies as a function of frequency.

IV. For a fixed resistor and input voltage, I measure the magnitude of the output voltage as a function of frequency. From Eq. (1), I see that if I plot $|V_{in}|^2/|V_{out}|^2$ as a function of $(\omega/R)^2$, the slope is equal to L^2. Alternatively, if I plot the square-root factor on the right-hand-side of Eq. (2) as a function of (ω/R), the slope is equal to L.

2.2   Example 2

Here is a more challenging black box example with two elements:

I am working with a box that I know has two components.

I. I use the ohmmeter (HP 34401A) to measure the DC resistance, and I find that the resistance is very low. From this I determine that one of the elements connecting the input terminal and the output terminal is either a very small resistor or an inductor.

II. I place a resistor in series with the box, apply a sinusoidal voltage to the series combination and measure the voltage across the black box as shown on the right.

I notice that as I increase the frequency of the input voltage, the output voltage first increases, reaches a defined maximum and then decreases. That is, I see a resonance peak. Therefore, the box must contain an inductor in parallel to a capacitor (why?).
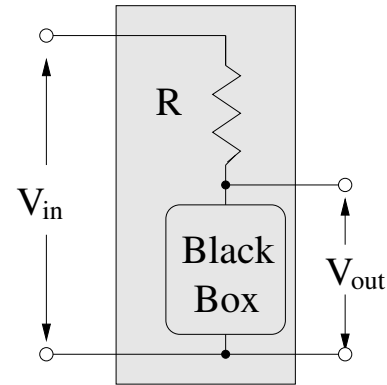
III. Since I now know there is an inductor in parallel with a capacitor inside the box, I can calculate what the output voltage must be. First I determine the impedance of the parallel inductor and capacitor:

$$Z_{\|} = \frac{Z_C \, Z_L}{Z_C + Z_L} = \frac{\frac{L}{C}}{i \left(\omega \, L - \frac{1}{\omega \, C} \right)} \tag{3}$$

Then I determine the complex-valued gain

$$\frac{V_{out}}{V_{in}} = \frac{Z_{\|}}{R + Z_{\|}} = \frac{\frac{L}{C}}{i \left(\omega \, L - \frac{1}{\omega \, C} \right)} \, \frac{1}{R + \frac{L/C}{i \left(\omega \, L - \frac{1}{\omega \, C} \right)}} = \frac{\frac{L}{C}}{i R \left(\omega \, L - \frac{1}{\omega \, C} \right) + \frac{L}{C}} \tag{4}$$

to obtain the square of the voltage-gain's magnitude.

$$\frac{|V_{out}|^2}{|V_{in}|^2} = \frac{\left(\frac{L}{C} \right)^2}{R^2 \left(\omega \, L - \frac{1}{\omega \, C} \right)^2 + \left(\frac{L}{C} \right)^2} \tag{5}$$

Can we massage Eq. (5) to yield a straight line plot? Possible? One solution to the problem is to use a nonlinear fitting routine on our data in order to estimate L and C in Eq. (5). Another approach is to instead determine L and C based on the measurement of the resonance frequency and the width of the resonance peak (no data fitting required).

The resonance frequency (also called center frequency) occurs where $|V_{out}|^2/|V_{in}|^2$ is maximum or, alternatively, where $|V_{in}|^2/|V_{out}|^2$ is minimum. This frequency can be measured very precisely and we can also determine its value theoretically. Although the answer is easy to guess, let us be systematic and determine where $|V_{in}|^2/|V_{out}|^2$ has zero slope:

$$0 = \frac{d}{d\omega} \left(\frac{R^2 \left(\omega \, L - \frac{1}{\omega \, C} \right)^2 + \left(\frac{L}{C} \right)^2}{\left(\frac{L}{C} \right)^2} \right) = \left(\frac{RC}{L} \right)^2 2 \left(\omega \, L - \frac{1}{\omega \, C} \right) \left(L + \frac{1}{\omega^2 \, C} \right). \tag{6}$$

There is only one real-valued solution:

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}} \tag{7}$$

Measuring the resonance frequency allows us to determine the product LC. To obtain the value of L and C separately we need a second measurement. One option is to determine the full-width-half-maximum (FWHM) value of the resonance peak by measuring the two frequencies where the square of the voltage-gain-amplitude has decreased by a factor of 2.

Theoretically, we need to solve

$$\frac{1}{2} = \frac{\left(\frac{L}{C}\right)^2}{R^2 \left(\omega L - \frac{1}{\omega C}\right)^2 + \left(\frac{L}{C}\right)^2} \quad \Rightarrow \quad R^2 \left(\omega L - \frac{1}{\omega C}\right)^2 = \left(\frac{L}{C}\right)^2, \tag{8}$$

which is not hard. Taking the square root of Eq. (8) and keeping track of the two cases yields

$$R\left(\tilde{\omega} L - \frac{1}{\tilde{\omega} C}\right) = \left(\frac{L}{C}\right) \qquad\qquad R\left(\omega L - \frac{1}{\omega C}\right) = -\left(\frac{L}{C}\right)$$

$$\tilde{\omega}^2 - \frac{\tilde{\omega}}{RC} - \frac{1}{LC} = 0 \qquad\qquad \omega^2 + \frac{\omega}{RC} - \frac{1}{LC} = 0$$

$$\tilde{\omega}_\pm = \frac{1}{2RC} \pm \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \qquad\qquad \omega_\pm = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}}$$

Relevant solution: Relevant solution:

$$\tilde{\omega}_+ = \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \qquad\qquad \omega_+ = -\frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}}$$

Thus, the theoretical $|V_{in}|^2/|V_{out}|^2$ resonance-peak width at half-maximum is

$$\Delta\omega = 2\pi\Delta f = \tilde{\omega}_+ - \omega_+ = \frac{1}{RC}. \tag{9}$$

Since we can measure R directly, we can determine the value of the capacitance C by measuring $\Delta\omega$, and the value of L by measuring the resonance frequency and using Eq. (7).

IV. The data is shown in Fig. 2.2. We measured

$$R = 1050\,\Omega \qquad \Delta f = 410\,\text{Hz} \qquad f_0 = 1410\,\text{Hz}. \tag{10}$$

Using Eq. (7) and Eq. (9) we obtain

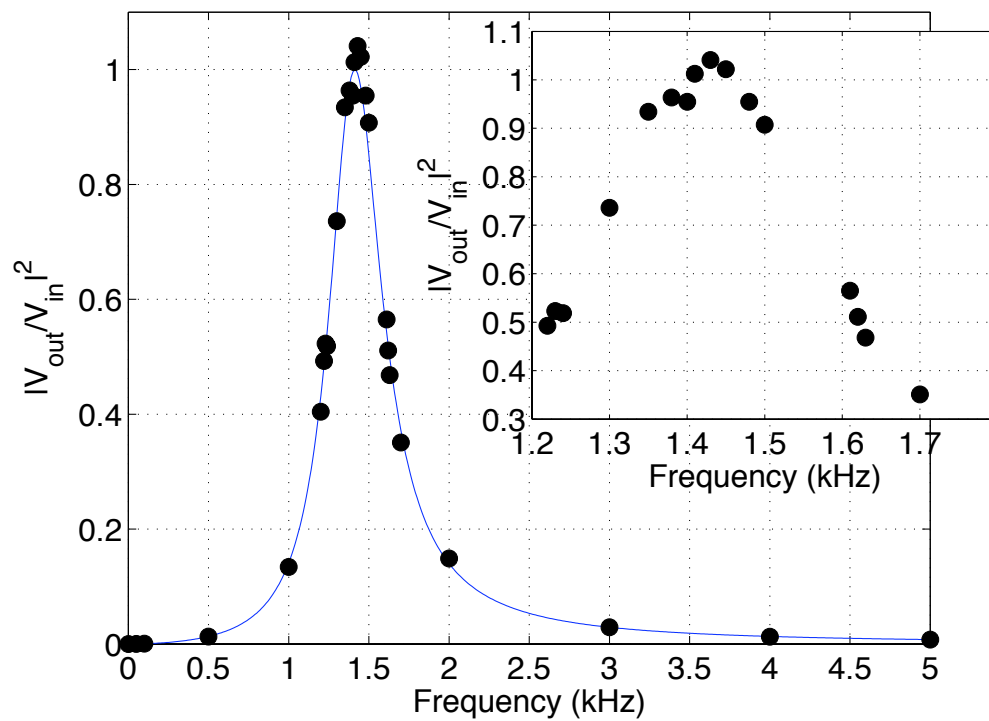$$C = 380\,\text{nF} \qquad L = 33\,\text{mH}. \tag{11}$$

Figure 4: Measured gain (voltage-gain magnitude to the second power) as a function of frequency. A clear resonance peak is seen. The circles are data points, the line is Eq. (5) with the estimated values of C and L. The inset shows a zoom of the peak from which we read off $\Delta f \approx 400\,Hz$ and $f_0 \approx 1410\,Hz$.