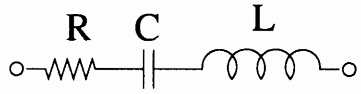
Name: ____________________________

There are **three questions** to complete.

**1.** Determine the magnitude $|Z|$ of the total impedance $Z$ of the following circuit




What are the limits on your expression for $|Z|$ as the angular frequency $\omega$ approaches 0 and $\infty$? Can you explain these limits intuitively?

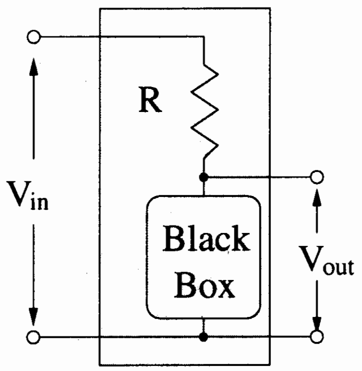
$$Z = Z_R + Z_C + Z_L = R + \frac{1}{i\omega C} + i\omega L = R + i\left(\omega L - \frac{1}{\omega C}\right)$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

$\omega \to 0$ $\quad |Z| \to \infty$ capacitor blocks current flow

$\omega \to \infty$ $\quad |Z| \to \infty$. inductor chokes off the current flow

**2.**




Plot qualitatively the expected behavior of $V_{out}$ as a function of the input frequency $f$ [$v_{in} = V_{in}\cos(2\pi f t)$] if the black box contains

(a) a resistor,

(b) an inductor,

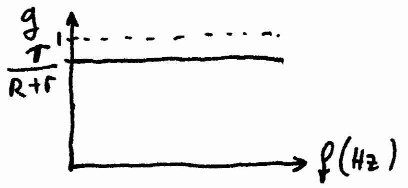
(c) a capacitor,

(d) an inductor and capacitor in parallel,

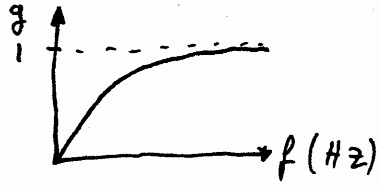
(e) an inductor and capacitor in series.

In general, the gain is $g = \frac{V_{out}}{V_{in}} = \left|\frac{Z}{R+Z}\right|$. For fixed $V_{in}$, $g \propto V_{out}$. Plot $g$, $g \in [0,1]$

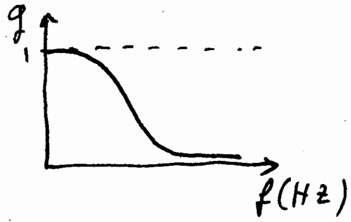
a) $Z = r$ $\quad g = \frac{r}{R+r}$




b) $Z = i\omega L$ $\quad g = \left|\frac{i\omega L}{R+i\omega L}\right| = \frac{\omega L}{\sqrt{R^2+\omega^2 L^2}}$

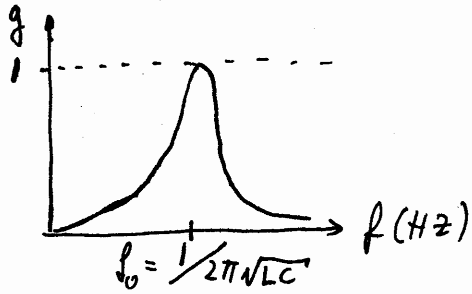



c) $g = \left|\frac{1/i\omega C}{R + 1/i\omega C}\right| = \left|\frac{1}{i\omega RC+1}\right| = \frac{1}{\sqrt{\omega^2 R^2 C^2 + 1}}$



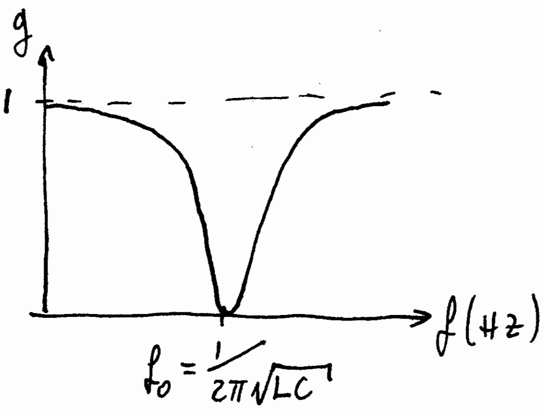

d) $Z = \frac{Z_L \cdot Z_C}{Z_L + Z_C} = \frac{i\omega L \cdot \frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}} = (-i)\frac{L/C}{(\omega L - \frac{1}{\omega C})}$

| | $\omega \to 0$ | $\omega \to \infty$ | $\omega \to \frac{1}{\sqrt{LC}}$ |
|---|---|---|---|
| $\lvert Z\rvert = \left\lvert\frac{L/C}{\omega L - \frac{1}{\omega C}}\right\rvert$ | 0 | 0 | $\infty$ |
| $g = \left\lvert\frac{Z}{R+Z}\right\rvert$ | 0 | 0 | 1 |

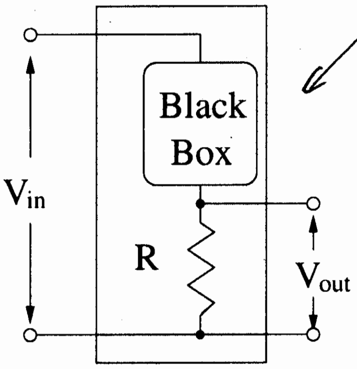



e) $Z = Z_L + Z_C = i\omega L + \frac{1}{i\omega C} = i\left(\omega L - \frac{1}{\omega C}\right)$

| | $\omega \to 0$ | $\omega \to \infty$ | $\omega \to \frac{1}{\sqrt{LC}}$ |
|---|---|---|---|
| $\lvert Z\rvert = \left\lvert\omega L - \frac{1}{\omega C}\right\rvert$ | $\infty$ | $\infty$ | 0 |
| $g = \left\lvert\frac{Z}{R+Z}\right\rvert$ | 1 | 1 | 0 |



**3.**




note, the B.B. is on top and $V_{out}$ is measured across R!

For a black box containing a capacitor $C$ and an inductor $L$ that are connected in series, derive the resonance frequency $f_0$ and the FWHM (full-width-half-maximum) value $\Delta f$ of the resonance peak that is seen when plotting $|v_{out}|^2/|v_{in}|^2 = V_{out}^2/V_{in}^2$. Give your answer in terms of $L, C$, and $R$.

$$Z = Z_L + Z_C = i\omega L + \frac{1}{i\omega C} = i\left(\omega L - \frac{1}{\omega C}\right)$$

$$\frac{|v_{out}|^2}{|v_{in}|^2} = \left|\frac{R}{R+Z}\right|^2 = \frac{R^2}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

1) Find resonance freq. $f_0$ :

Method A: Guess that $\omega_0 = \frac{1}{\sqrt{LC}}$ . Then $\frac{|v_{out}|^2}{|v_{in}|^2} = \frac{R^2}{R^2+0} = 1$, which we know is the maximum possible gain for a passive-element circuit such as this one. So our guess must be correct.

Method B: Find max. of $(V_{out}/V_{in})^2$, or min. of $(V_{in}/V_{out})^2$, formally:

$$0 = \frac{d}{d\omega}\left(V_{in}/V_{out}\right)^2 = \frac{d}{d\omega}\left(1 + \frac{1}{R^2}\left[\omega L - \frac{1}{\omega C}\right]^2\right) = \frac{2}{R}\left(\omega L - \frac{1}{\omega C}\right)\left(L + \frac{1}{\omega^2 C}\right)$$

$\hookrightarrow$ $\omega_0 = \frac{1}{\sqrt{LC}}$ only solution in $\mathbb{R}$

$$\Rightarrow \boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

2) Find FWHM:

$$\frac{1}{2} = \left(\frac{V_{out}}{V_{in}}\right)^2 = \frac{R^2}{R^2 + (\omega L - \frac{1}{\omega C})^2} \longrightarrow \text{Solve } \left(\omega L - \frac{1}{\omega C}\right)^2 = R^2$$

That is: $\omega_\pm L - \frac{1}{\omega_\pm C} = \pm R \quad \Big| \cdot \frac{\omega_{+-}}{L}$

$$\hookrightarrow \omega_\pm^2 \mp \omega_\pm \frac{R}{L} - \frac{1}{LC} = 0$$

The two real and positive solutions are (one for $\omega_+$ and one for $\omega_-$)

$$\omega_+ = +\frac{R}{2L} + \sqrt{\frac{R^2}{2L} + \frac{1}{LC}} \qquad \omega_- = -\frac{R}{2L} + \sqrt{\frac{R^2}{2L} + \frac{1}{LC}}$$

$$\hookrightarrow \Delta\omega = \omega_+ - \omega_- = \frac{R}{L}$$

$$\boxed{\Delta f = \frac{R}{2\pi L}}$$