Laboratory IV: Filter

Reading: Simpson Chapter 11.1 - 11.6

1 Introduction

In this lab we will explore some simple analog electronic filters. Filters are often used to separate a desired analog signal from the noise-background and they are essential to avoid an effect called aliasing when interfacing an experiment with a computer. For a given filter, its *passband* is the frequency range over which signals are allowed to pass from the input to the output, whereas its *stopband* is the frequency range over which input signals are significantly attenuated. We may therefore distinguish the following basic filter types shown in Fig. 1

- Low-pass filter – Low frequencies are passed and high frequencies are attenuated.
- High-pass filter – High frequencies are passed and low frequencies are attenuated.
- Band-pass filter – Only frequencies in a chosen frequency band are passed.
- Band-stop (Notch) filter – Only frequencies in a chosen frequency band are attenuated.
- All-pass filter – All frequencies are passed, but the phase of the output is modified.

The steepness of the decrease of gain as a function of frequency in the stopband region is an important characteristic of a filter. This decrease – that is, the change in attenuation per unit frequency – is referred to as the filter *rolloff*.

Passive filters, such as RC low-pass and high-pass filters, allow only for a frequency-dependent loss, *i.e.* the output voltage is always less than the input voltage. Additionally, their rolloff is relatively modest. Active filters permit a frequency-dependent gain as well as loss, and can have a steeper roll-off. Nowadays, single integrated circuits are available that will implement many of the sophisticated filters such as the Butterworth, Bessel and Chebyshev filters described in Simpson. However, in this course we will focus on simple active filters that can be build using operational amplifiers.

2 Jargon

Passband : Frequency range over which signals pass from the input to the output of a filter.

Stopband : Frequency range over which input signals are significantly attenuated.

Rolloff : Decrease of gain as a function of frequency in the stopband region.

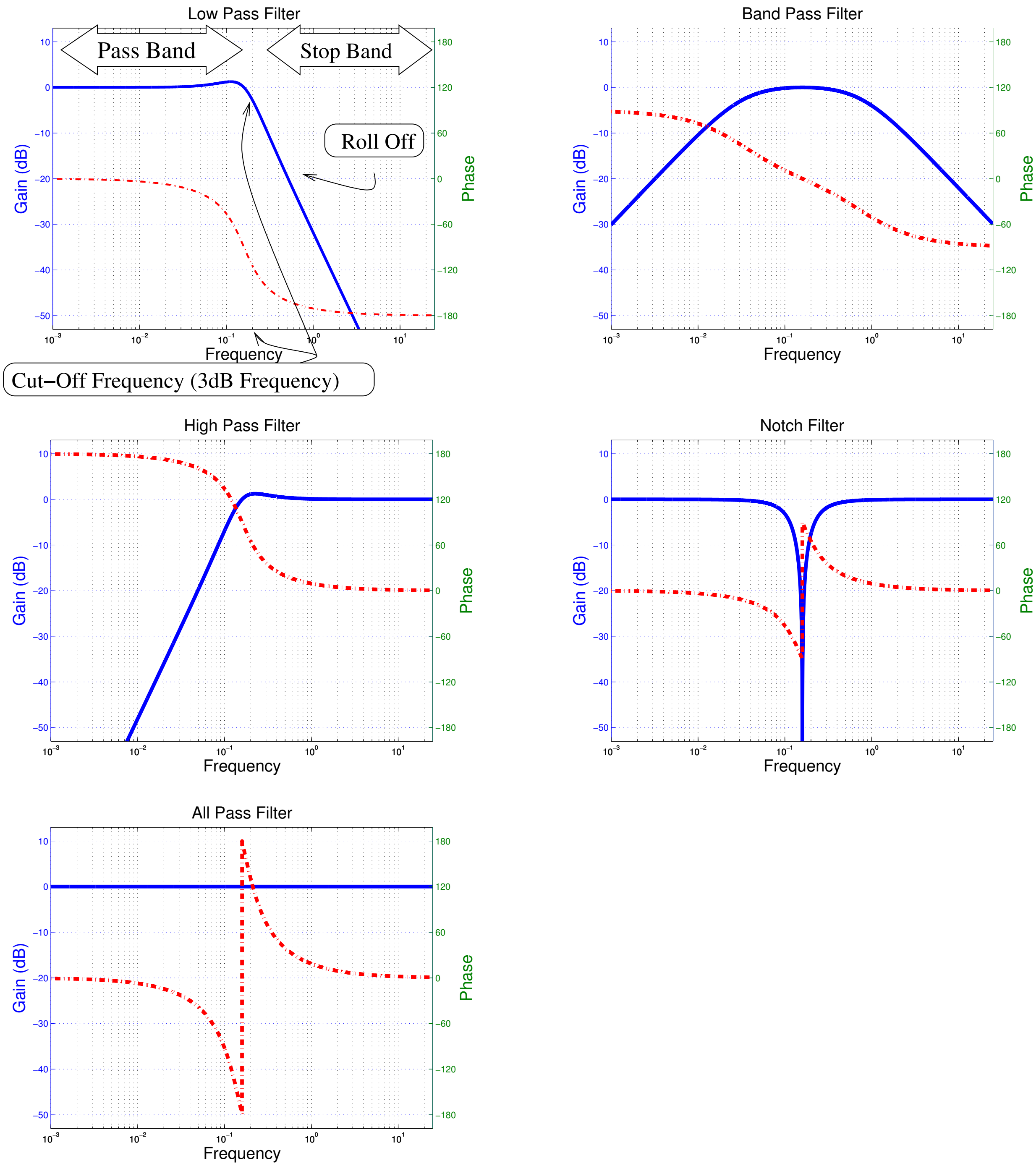

Figure 1: Examples illustrating the basic filter types. Shown are the gain (solid) and phase (dash-doted) versus frequency (arb. units).

dB (decibel) : A ratio of powers can be expressed in decibel by computing

$$\text{power ratio in decibel } = 10\ \log_{10}\left(\frac{P}{P_0}\right) \tag{1}$$

Where P is the measured power of interest and P_0 is a common reference power that has to be specified. Since power is proportional to the square of the voltage $P \propto V^2$, a voltage ratio in decibels is computed as follows

$$\text{voltage ratio in decibel } = 10\ \log_{10}\left(\frac{V^2}{V_0^2}\right) = 10\ \log_{10}\left(\left[\frac{V}{V_0}\right]^2\right) = 20\ \log_{10}\left(\frac{V}{V_0}\right) \tag{2}$$

A voltage gain is nothing but a voltage ratio. A factor of $\sqrt{2}$ decrease in voltage (factor 2 in power), meaning the output voltage has an amplitude that is $1/\sqrt{2}$ times the amplitude of the input voltage, $V_{out}/V_{in} = 1/\sqrt{2}$, implies a gain of -3dB, because dB = $20\ \log_{10}(1/\sqrt{2})$ = -3.01. If V_{out}/V_{in}= 1/2 , then dB = 20 log(1/2) = -6.02. If V_{out}/V_{in}= 1/10 , then dB = 20 log(1/10) = -20. If V_{out}/V_{in}= 1/100 , then dB = 20 log(1/100) = -40.

20 dB per decade A function $V(f)$ which is proportional to $1/f$ is said to "fall off" (or "roll off") at the rate of 20 dB per decade. That is, for every factor of 10 in frequency f (every "decade"), the amplitude drops 20 dB because

$$20\ \log_{10}\left(\frac{V(10\,f)}{V(f)}\right) = 20\ \log_{10}\left(\frac{1/(10\,f)}{1/f}\right) = 20\ \log_{10}\left(\frac{1}{10}\right) = -20 \tag{3}$$

6 dB per octave A function $V(f)$ which is proportional to $1/f$ is said to fall off 6 dB per octave. That is, for every factor of 2 in the frequency f (every "octave"), the voltage drops close to 6 dB. Thus, 6 dB per octave is the same thing as 20 dB per decade.

***H* - the transfer function** Although Simpson uses a different notation, the standard notation for the transfer function is H. A given $H(\omega)$ fully characterizes the gain and phase of a linear filter. In our context

$$H(\omega) = \frac{V_{out}}{V_{in}}. \tag{4}$$

The filter gain at a given frequency is simply the magnitude of the transfer function

$$\text{gain at } \omega = |H(\omega)| = \frac{|V_{out}|}{|V_{in}|}. \tag{5}$$

f_{3dB} - Corner Frequency The frequency at which the voltage gain falls to $1/\sqrt{2}$ of its maximum value ($20\ \log_{10}(1/\sqrt{2}) = -3.01$).

3 Passive Filters

RC filters are the simplest filters one can build (see Fig. 2). In these circuits the relationship between the magnitudes of the output voltage and the input voltage, *i.e.* the gain, is given by the voltage divider formula:

$$|H_{LP}(\omega)| = \left|\frac{V_{out}(\omega)}{V_{in}}\right| = \left|\frac{\frac{1}{i\omega C}}{R+\frac{1}{i\omega C}}\right| = \left|\frac{1}{i\omega RC+1}\right| = \frac{1}{\sqrt{(\omega RC)^2+1}} \tag{6}$$

$$|H_{HP}(\omega)| = \left|\frac{V_{out}(\omega)}{V_{in}}\right| = \left|\frac{R}{R+\frac{1}{i\omega C}}\right| = \left|\frac{i\omega RC}{i\omega RC+1}\right| = \frac{\omega RC}{\sqrt{(\omega RC)^2+1}} \tag{7}$$

From these equations we see, that for the low pass filter the gain $|H| < 1$ for all nonzero frequencies and $\max|H| = H(0) = 1$, that is, the output amplitude is always less than or equal to the input amplitude. This is generally true for passive filters, the maximum gain $|H|$ is one and for almost all frequencies $|H| < 1$. The 3 dB-frequency (corner frequency) is given by $f_{3dB} = 1/RC$ for both the high-pass and low-pass filter (check this yourself).

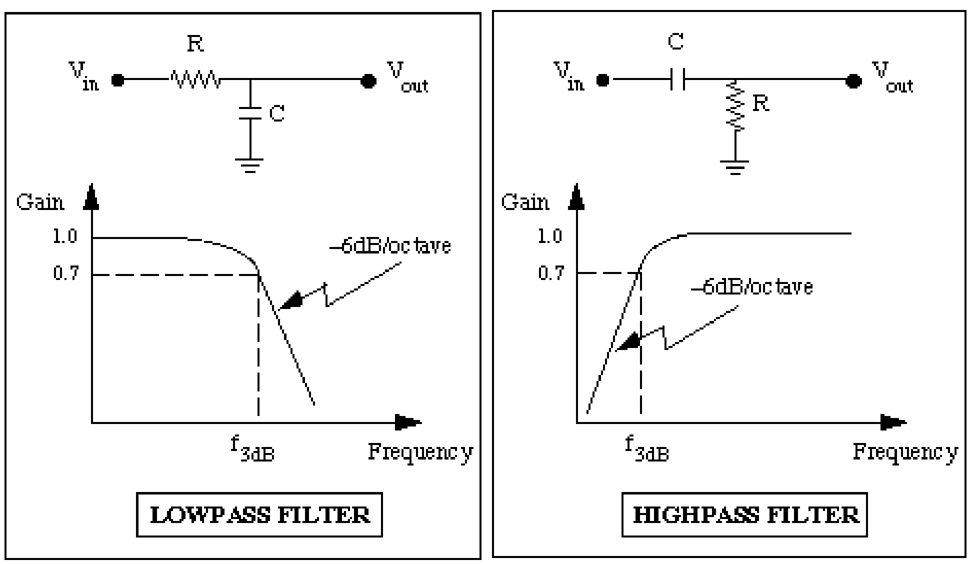

Figure 2: RC low pass and high pass filter (Adapted from John Essick)

The low-pass RC filter has a mild rolloff of "-6 decibels per octave" because in the stopband (for $f \gg f_{3dB}$) the gain is proportional to $1/f$

$$|H(\omega)| \approx \frac{1}{\omega RC}, \qquad \text{for } f \gg f_{3dB} \text{ (, which is equivalent to the condition } \omega RC \gg 1).$$

For many filter applications, it is desirable to have a steep rolloff, *i.e.*, to have an abrupt transition between the filter's passband and stopband. One way to achieve this is to cascade two or more RC filters, one after another, to increase the sharpness of the rolloff, each stage adding another 6 dB per octave. In principle, we could build high-quality filters in precisely this manner. However, passive filters have a couple of drawbacks. Because they provide no gain, only a loss, signals can quickly get reduced in amplitude until they are lost in the noise. Also, their input impedance is typically not as large as we might like, so they "load down" the previous stage in the circuit. Furthermore, their output impedance is not very small, so they are easily "loaded down" by the next stage. The straightforward solution is to place op amps before and after the RC filter, to buffer it and add gain. But, at this point it would be just as easy to use an "active filter."

4 Active Filters

Active filters involve components that can add energy to the system, for example operational amplifiers. The design of active filters is a deep topic and is studied widely in engineering because filters can be designed to do an amazing amount of signal manipulation, analysis, and processing. What happens on a basic level is that an input signal is put into a filter, the filter reacts in some way and produces an output signal. Thus, from an input-output perspective, the filter transforms (or processes) the signal. Engineers are then interested in knowing how to design filters such that they perform a desired transformation. Traditional filters, such as the ones discussed here, are linear devices and can therefore be analyzed and understood completely using the tools of linear systems theory. However, this theory is beyond the scope of this course. Fortunately, we can discuss the basic considerations using simple circuits that can be analyzed without too much theory.

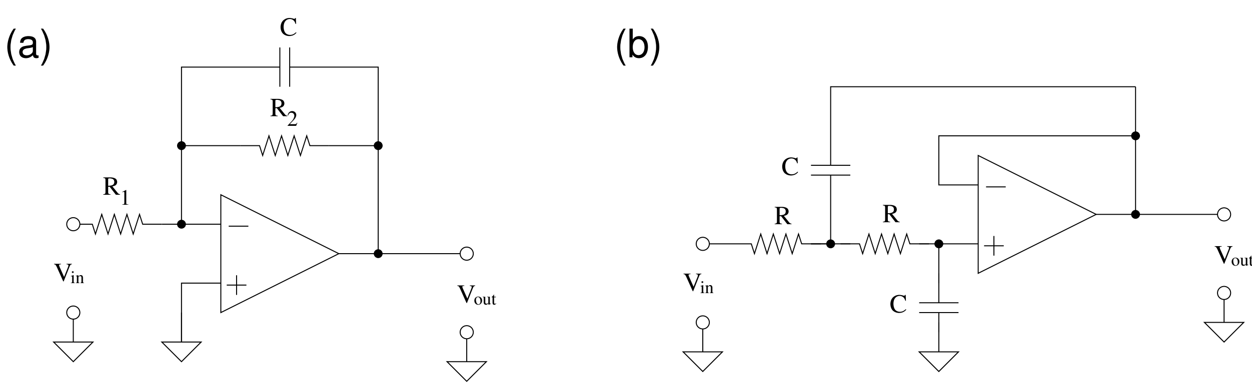

Figure 3: (a) First order active low-pass filter. (b) Sallen-Key second order low-pass filter.

The first-order low-pass active RC filter shown in Fig. 3(a) has a gain

$$|H(\omega)| = \left| \frac{V_{out}(\omega)}{V_{in}} \right| = \frac{R_2}{R_1} \frac{1}{\sqrt{1 + (\omega\, R_2\, C)^2}}. \tag{8}$$

It is seen from this equation that the rolloff is -20dB per decade (-6 dB per octave). The

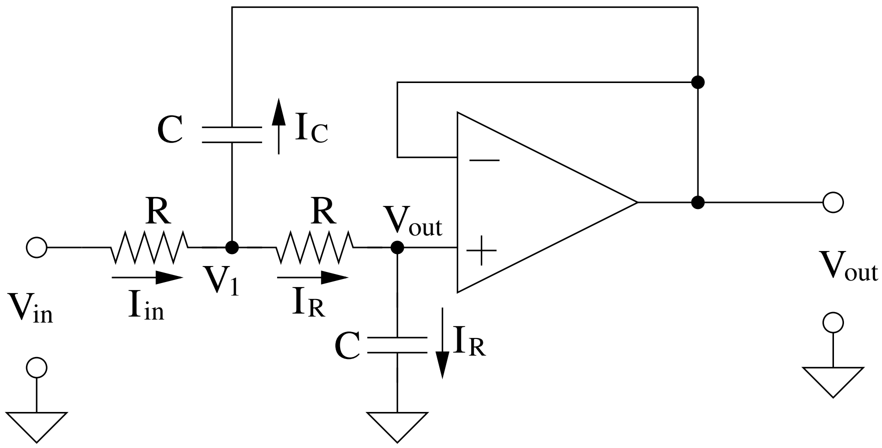

Figure 4: Sallen Key low pass filter.

Sallen-Key active filter shown in Fig. 3(b) is a second-order low-pass filter with a gain of

$$|H(\omega)| = \left|\frac{V_{out}(\omega)}{V_{in}}\right| = \frac{1}{1+(\omega\, RC)^2}. \tag{9}$$

and a rolloff of -40dB per decade (-12 dB per octave). [Both Eq. (8) and Eq. (9) can be derived using the Golden Rules for operational amplifiers. Try it!]

5 Second-Order Low-Pass Filters

Last week's lab handout we showed the for the first-order low pass filter

$$t_{63\%} = \tau_{RC} = RC = \frac{1}{2\pi f_{3dB}} \tag{10}$$

and

$$\text{R.T.} = \ln 9 \cdot \tau_{RC} = \frac{0.35}{f_{3dB}} \tag{11}$$

holds. These equations relate the rise time and $t_{63\%}$, which are obtained through a time-domain measurement, to the bandwidth (f_{3dB}), which is obtained through a frequency-domain measurement.

We would like to determine theoretically, what the equivalent relations are for the Sallen-Key second order low-pass filter.

5.1 Differential Equation

For the Sallen-Key circuit depicted in Fig. 4b) and reproduced in Fig. 4 we may use Kirchhoff's laws and the Golden Rules for op-amps to obtain

$$\left.\begin{aligned}\frac{I_R}{C} &= \frac{d}{dt}v_{out} \\ I_R &= \frac{(v_1 - v_{out})}{R}\end{aligned}\right\} \quad \left(RC\,\frac{d}{dt} + 1\right) v_{out} = v_1 \quad \text{or} \quad (v_1 - v_{out}) = RC\,\frac{d}{dt}v_{out} \tag{12}$$

In addition we have the relations

$$I_{in} = \frac{v_{in} - v_1}{R} \tag{13}$$

$$I_C = C\frac{d}{dt}\left(v_1 - v_{out}\right) \tag{14}$$

$$I_{in} = I_C + I_R \tag{15}$$

Make sure you understand how to obtain above relations (Eq. 12 -15) from the circuit diagram. We may now derive the differential equation by multiplying Eq. 15 with R and plugging in Eq. 12 -14.

$$R\,I_{in} = R\,I_C + R\,I_R \tag{16}$$

$$[v_{in} - v_1] = \left[RC\frac{d}{dt}\left(v_1 - v_{out}\right)\right] + \left[RC\frac{d}{dt}v_{out}\right] \tag{17}$$

$$\left[v_{in} - \left(RC\,\frac{d}{dt} + 1\right) v_{out}\right] = \left[RC\frac{d}{dt}\left(RC\,\frac{d}{dt}v_{out}\right)\right] + \left[RC\frac{d}{dt}v_{out}\right] \tag{18}$$

Introduce as before

$$\tau_{RC} = RC \tag{19}$$

and simplify to get

$$\left(\tau_{RC}^2\frac{d^2}{dt^2} + 2\tau_{RC}\frac{d}{dt} + 1\right) v_{out} = v_{in} \tag{20}$$

or equivalently

$$\left(\tau_{RC}\frac{d}{dt} + 1\right)\left(\tau_{RC}\frac{d}{dt} + 1\right) v_{out} = v_{in} \tag{21}$$

We find that the second-order Sallen-Key filter connects the input and output via a second order linear ordinary differential equation.

5.2 Frequency Domain

You should be able to derive the transfer function $H(\omega)$, gain $|H(\omega)|$ and be able to show that the gain has a maximum of 1 at $\omega = 0$ and a 3dB point at

$$f_{3dB} = \frac{\sqrt{\sqrt{2} - 1}}{2\pi RC} \tag{22}$$

(Try it!)

Note that this corner frequency is different from the corner frequency of a first-order low pass filter.

5.3 Time Domain

Any solution to a linear ordinary differential equation is the sum of two parts: the solution to the homogeneous equation and the particular solution. The general solution to the homogeneous equation, the equation with $v_{in} = 0$, can be easily guessed because we showed last week that $C\ \exp(-(t-t_0)/\tau_{RC})$ is the solution to

$$\left(\tau_{RC}\frac{d}{dt}+1\right)v_{out}=0 \tag{23}$$

where C is a constant of integration depending on the initial value. The homogeneous second order equation

$$\left(\tau_{RC}\frac{d}{dt}+1\right)\left(\tau_{RC}\frac{d}{dt}+1\right)v_{out}^h=0 \tag{24}$$

has a solution consisting of two linearly independent terms and therefore also has two constants of integration. You may check that the solution is

$$v_{out}^h = C_1\, e^{-(t-t_0)/\tau_{RC}} + C_2\,(t-t_0)\, e^{-(t-t_0)/\tau_{RC}} \tag{25}$$

As in last week's derivation for the first order RC filter, the homogeneous solution decays away quickly and does not matter for most measurements. It therefore does not need to be taken into account. Formally this is again achieved by using the limit $t_0 \to -\infty$.

What we are interested in is the particular solution for a step input

$$\left(\tau_{RC}\frac{d}{dt}+1\right)\left(\tau_{RC}\frac{d}{dt}+1\right)v_{out}=\theta(t)\,V_{max}. \tag{26}$$

For $t < 0$, when the right hand side is zero, the particular solution is $v_{out} = 0$. For $t > 0$ the right hand side is $\theta(t) = 1$ and we can find the solution from Eq. 25. You may check that

$$\frac{v_{out}}{V_{max}} = 1 - e^{-t/\tau_{RC}} - \frac{t}{\tau_{RC}}\, e^{-t/\tau_{RC}} \qquad t > 0 \tag{27}$$

is the particular solution for a step input. Having this solution at hand we can now calculate the rise time and indeed the time it takes the solution to reach any desired voltage level between 0 and V_{max}. Of particular interest are

$$x = 1 - \left(1+\frac{t}{\tau_{RC}}\right) e^{-t/\tau_{RC}} \qquad \text{with } x = \frac{v_{out}}{V_{max}} = 10\%,\ 63\%,\ 90\% \tag{28}$$

Solving for t_x is tricky. You can do it numerically or by using the Lambert W - function

$$t_x = \tau_{RC}\cdot\left[-1 - W_{-1}\left(\frac{x-1}{e}\right)\right] \tag{29}$$

where W_{-1} is the -1 branch of the Lambert W function. In particular the rise time and $t_{63\%}$-time are

$$R.T. = t_{90\%} - t_{10\%} = \tau_{RC} \cdot 3.36 \qquad \longleftarrow \quad \text{Note that } 3.36 \neq \ln 9 \tag{30}$$

$$t_{63\%} = \tau_{RC} \cdot 2.15 \qquad \longleftarrow \quad \text{Note that } \tau_{RC} \neq t_{63\%} \tag{31}$$

5.4 Summary

From this little calculation we learn that for a second-order Sallen-Key filter the relation between the RC time-constant, $\tau_{RC} = RC$, and the R.T. as well as with the time to reach 0.63 of V_{max}, $t_{63\%}$, changes. In addition, the relation determining the corner frequency, Eq. 22, is new. We may combine these result to obtain

$$\text{R.T.} = \tau_{RC} \cdot 3.36 = RC \cdot 3.36 = \frac{\sqrt{\sqrt{2}-1}}{2\pi f_{3dB}} \cdot 3.36 = \frac{0.34}{f_{3dB}} \tag{32}$$

For reasons that are either very deep and not obvious to me or just due to an accidental coincidence, the equation that relates the rise time to the bandwidth (f_{3dB}) is numerically quite close to the first order low pass filter case (Eq. (9.17) in Simpson pg. 384 [2]). Again, this relation is very useful because it allows one to compare results from a time-domain measurement (R.T.) to a frequency-domain measurement (f_{3dB}).

Table 1: Comparison of the results of an RC first-order low-pass filter and a Sallen-Key second-order low-pass filter

First-Order RC Low-Pass Filter	Second-Order Sallen-Key Low-Pass Filter
$\tau_{RC} := RC$	$\tau_{RC} := RC$
$f_{3dB} = (2\pi RC)^{-1}$	$f_{3dB} = \sqrt{\sqrt{2}-1}\,(2\pi RC)^{-1}$
$t_{63\%} = \tau_{RC}$	$t_{63\%} = 2.15\,\tau_{RC}$
$R.T. = \ln 9\ \tau_{RC} = 2.20\ \tau_{RC}$	$R.T. = 3.36\ \tau_{RC}$
$R.T. = 0.35\ f_{3dB}^{-1}$	$R.T. = 0.34\ f_{3dB}^{-1}$

References

[1] L. Illing, *Lab III - Handout*, unpublished notes for Physics 331, Reed College, Fall 2008.

[2] R. E. Simpson, *Introductory Electronics for Scientists and Engineers*, 2nd Ed., Allyn and Bacon, Inc. (Boston, 1987).