The Fourier Transform

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1 Fourier Series

A periodic function $f(x)$ defined on $x \in [0, a]$ has a Fourier series expansion:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m \, e^{\frac{i\,2\,\pi\,m\,x}{a}} \tag{1}$$

if the Dirichlet conditions hold:

- The function $f(x)$ is periodic with period a (assumed in definition).
- $f(x)$ has a finite number of minima, maxima and discontinuities on $[0, a]$.
- $\int_0^a |f(x)| \, dx$ is finite.

What we mean by "has a Fourier series expansion" is: The series on the right converges to $f(x)$ at all points $x \in [0, a]$ where $f(x)$ is continuous, and converges to the midpoint of any discontinuities in $f(x)$.

When the Fourier series exists, we can access the coefficients of the expansion by exploiting:

$$\int_0^a e^{-\frac{i\,2\,\pi\,m\,x}{a}} \, e^{\frac{i\,2\,\pi\,n\,x}{a}} \, dx = a \, \delta_{mn}. \tag{2}$$

Then

$$c_m = \frac{1}{a} \int_0^a e^{-\frac{i\,2\,\pi\,m\,x}{a}} \, f(x) \, dx. \tag{3}$$

1.1 Example

Take the discontinuous function:

$$f(x) = \begin{cases} x & x < \frac{1}{2}\,a \\ x+1 & x > \frac{1}{2}a \end{cases}\ . \tag{4}$$

According to (3), we have:

$$\begin{aligned} c_m &= \frac{1}{a}\left(\int_0^{\frac{1}{2}\,a} x\,e^{-\frac{i\,2\,\pi\,m\,x}{a}}\,dx + \int_{\frac{1}{2}\,a}^{a} (x+1)\,e^{-\frac{i\,2\,\pi\,m\,x}{a}}\,dx\right) \\ &= \frac{1}{a}\left(\int_0^{a} x\,e^{-\frac{i\,2\,\pi\,m\,x}{a}} + \int_{\frac{1}{2}\,a}^{a} e^{-\frac{i\,2\,\pi\,m\,x}{a}}\,dx\right). \end{aligned} \tag{5}$$

Using integration-by-parts on the first term, and integrating the second, I get:

$$c_m = \frac{i\,a}{2\,\pi\,m} - \frac{i\,(-1+(-1)^m)}{2\,\pi\,m}. \tag{6}$$

There is clearly a special case here, at $m = 0$, and this sets an overall constant for the function $f(x)$. If we define the approximate series:

$$f_n(x) = 1 + \sum_{m=-n}^{n} c_m\,e^{\frac{i\,2\,\pi\,m\,x}{a}}, \tag{7}$$

with $m \neq 0$, we can get a sense for the "convergence". A few values for n are shown in Figure 1.

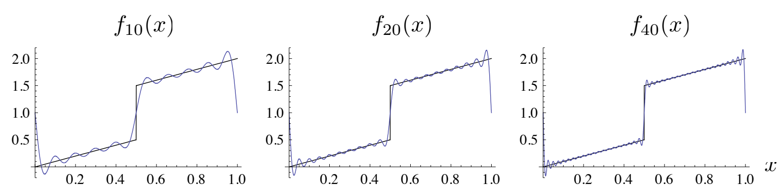

Figure 1: Approximate Fourier Series for the function defined in (4), $f(x)$ itself is shown in black.

Finally, note that we can, clumsily, introduce the c_m directly into the expansion:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{a}\left(\int_0^a e^{-\frac{i\,2\,\pi\,n\,x}{a}}\,f(x)\,dx\right) e^{\frac{i\,2\,\pi\,n\,x}{a}} \tag{8}$$

2 Fourier Transform

One way to think of the continuous Fourier transform is to consider our function $f(x)$ to be periodic with $a \to \infty$. This allows us to make a connection with the Fourier series, but does not count as a proof of existence, uniqueness or anything else. The following is for motivation only, my goal is to give us a way to talk about the Fourier transform, not rigor (for now).

Our first move will be to symmetrize the interval – suppose we define $f(x)$ on $x \in [-a, a]$, this changes almost nothing – we can rewrite (8) to reflect the change:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\,a} \left(\int_{-a}^{a} e^{-\frac{i\,\pi\,n\,x}{a}} \, f(x)\, dx \right) e^{\frac{i\,\pi\,n\,x}{a}}. \tag{9}$$

Let $p_n \equiv \frac{\pi\,n}{a}$, so that we can think of a "grid" of values p_n indexed by the integer n. The spacing of this grid is $p_{n+1} - p_n = \frac{\pi}{a} \equiv \Delta p$. The eventual "limit" $a \longrightarrow \infty$ will be taken by sending $\Delta p \longrightarrow 0$, giving us a continuum of values p. For now, we have

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{\Delta p}{2\,\pi} \left(\int_{-\pi/\Delta p}^{\pi/\Delta p} e^{-i\,p_n\,x} \, f(x)\, dx \right) e^{i\,p_n\,x}. \tag{10}$$

Now we can think about the limit. We know that an integral can be approximated by box-sums:

$$\int_{-A}^{A} g(p)\, dp \sim \sum_{n=-N}^{N} g(p_n)\, \Delta p. \tag{11}$$

The right-hand side can be thought of as the starting point in the definition of the integral (without the limit). What we have, in (10) is precisely such an expression, with:

$$g(p_n) = \frac{e^{i\,p_n\,x}}{2\,\pi} \int_{-\pi/\Delta p}^{\pi/\Delta p} e^{-i\,p_n\,x} \, f(x)\, dx \tag{12}$$

and N going to infinity. This would be the source of some fancy footwork in carefully taking the limit, but we are motivating only.

Squinting, now, we take $\Delta p \longrightarrow 0$, leaving us with a continuous variable $p_n \longrightarrow p$, and an integral over p:

$$f(x) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} e^{i\,p\,x} \left[\int_{-\infty}^{\infty} e^{-i\,p\,x} \, f(x)\, dx \right] dp. \tag{13}$$

It is from here that we arbitrarily factor the $\frac{1}{2\,\pi}$ and define:

$$\begin{aligned} \tilde{f}(p) &= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-i\,p\,x} \, f(x)\, dx \\ f(x) &= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{i\,p\,x} \, \tilde{f}(p)\, dp. \end{aligned} \tag{14}$$

The pair above encapsulates the content of (13), and our interpretation of $\tilde{f}(p)$ as "coefficients" in the decomposition of $f(x)$ into $e^{i\,p\,x}$ comes from the fact that they naturally inherited the role of the c_m from the Fourier series, that is where they came from.