The Riemann Tensor

Lecture 13

Physics 411
Classical Mechanics II

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We have, so far, studied classical mechanics in tensor notation via the Lagrangian and Hamiltonian formulations, and the special relativistic extension of the classical L and (to a lesser extent) H. To proceed further, we must discuss a little more machinery. The key points to keep in mind is that both classical and SR mechanics relied on a free particle Lagrangian whose equations of motion amount to extremizing the distance along curves. In a sense, that is what force free mechanics *is*.

We have also seen that of the four forces in nature, only one is naturally suited to special relativity. Our goal now is to bring gravity into the picture. In general relativity (where the fundamental notion is that the laws of physics are the same in *any* frame, and that locally, every frame looks inertial), we eliminate the notion of force for gravitational interaction, and introduce, instead, curves whose extrema are precisely the physical motion implied by Newton's law of gravity in the appropriate limit. That is, we will be looking entirely at force-free mechanics, where the interesting physics is driven by the large-scale geometry of space-time.

So the goal is to find particular geometries in which geodesics look, far away from sources, like planetary orbits (for example). In a sense, the job is much simpler than an arbitrary classical mechanics problem – for motion in a provided "geometry", we need a Lagrangian that extremizes a general notion of length. The "test particle problem" is then very easy to formulate. Much more difficult is the connection between "geometry" and sources (we haven't even defined the sources of geometry). In order to nail down both of these ideas, we must be able to discuss curved spaces in some detail. There are a two natural questions we should start with: How is "geometry" described? How can we tell if a particular equation of motion is an artefact of a particular coordinate system or if there is really new physics in it? The answer to the first question, for us, will be "by the metric", and the answer to the second will be "using the Riemann tensor."

We specialize to Riemannian manifolds – these are a special class of points with labels that have the nice, physical property, that at any point, we can introduce a coordinate system in which space-time looks like Minkowski – an important feature given our earth-based laboratories (where space-time is described via the Minkowski metric). In such space-times, the metric defines a connection (for us, they will anyway), the connection defines the Riemannian curvature, and the Riemannian curvature is, by Einstein's equations, related to sources. So for Einstein's theory, the metric is enough[1].

As for the structure of space-time, we have already seen that the metric, even the Euclidean metric, can have very different forms while describing the same geometry (Pythagorean theorem holds in Cartesian, spherical and cylindrical coordinates, each of which has a different metric). The equations of motion (which we have already seen) for a test particle have terms which might look like effective forces in one coordinate system, but not in another (a "centrifugal barrier" is the classic example from mechanics). We do not want to be fooled – coordinate systems can't matter, only the intrinsic geometry of the space-time, and for this we have to have the Riemann tensor – there is no other way to tell if space-time is flat (Minkowski) or not (GR with sources).

13.1 Extremal Lengths

Going back to our notion of length, one of the defining features of Riemannian geometry is its line element – we know that the metric defines everything in our subset of these geometries, so the "distance" between two points always has the same quadratic structure:

$$ds^2 = dx^\mu \, g_{\mu\nu} \, dx^\nu. \tag{13.1}$$

This tells us, again, how to measure lengths given a particular position (via the coordinate dependence of $g_{\mu\nu}$) in space-time. How can we get an invariant measure of length along a curve? For a curve, parametrized by

[1]There are other geometric indicators available in general, but space-times with these more complicated structures (torsion is a famous example) do not produce theories of gravity that are "better" (predict more, more accurate, or easier to use) than GR itself.

$x^\mu(\tau)$, we can write the "small displacement" dx^μ as $\dot{x}^\mu \, d\tau$. Then the line element (squared) along the curve is given by

$$ds^2 = \dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu \, d\tau^2, \tag{13.2}$$

just as in classical mechanics (where $g_{\mu\nu}$ is the Euclidean metric) and special relativistic mechancis (with $g_{\mu\nu}$ the Minkowski metric).

As always, for a generic $g_{\mu\nu}$, the total "length" of the curve is obtained by integrating, we can take

$$S = \int_a^b \sqrt{\dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu} \, d\tau \tag{13.3}$$

or, equivalently, in arc length parametrization[2]

$$\tilde{S} = \int_a^b \dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu \, d\tau. \tag{13.4}$$

The point is, suppose we vary this w.r.t. x^μ:

$$\delta\tilde{S} = \left(2 \, \ddot{x}_\gamma + 2 \, g_{\gamma\alpha,\beta} \, \dot{x}^\alpha \, \dot{x}^\beta - g_{\alpha\beta,\gamma} \, \dot{x}^\alpha \dot{x}^\beta\right) = 0. \tag{13.5}$$

Or, as we have seen many times:

$$\boxed{\ddot{x}_\gamma + \Gamma_{\gamma\alpha\beta} \, \dot{x}^\alpha \, \dot{x}^\beta = 0.} \tag{13.6}$$

This is what we would call the "equation of motion for a free particle" in Euclidean space – a line. In our generic space (or as generic as we can get given the restriction of our study to metric spaces), the interpretation is the same.

Here again, we can see how important it is that we differentiate between coordinate choices and geometry – the trajectory of a free particle cannot depend on the coordinate system, but *must* depend on the geometry in which it defines an extremal curve.

[2] Arc length parametrization is just the usual proper time parametrization from special relativity in units where $c = 1$ – amounts to defining a unit tangent vector to the curve: $\sqrt{-\dot{x}^\mu \, g_{\mu\nu} \, x^\nu} = 1$.

13.2 Cross Derivative (in)Equality

The definition of the Riemann tensor comes when we ask how the parallel transport of vectors depends on the path we take. Note that in a Euclidean space (or Minkowski, for that matter), parallel transport of vectors is independent of path – we pick up a vector and move it parallel to itself. This is well-defined in these flat spaces, but for an aribtrary metric, parallel transport is an ODE: $f^{\alpha}{}_{;\beta}\,\dot{x}^{\beta} = 0$ for a contravariant vector field f^{α} and a particular curve with tangent $\dot{x}^{\beta}$.

We will see that interpretation of the Riemann tensor in a moment – for its definition, we turn to cross-derivative inequality for the covariant derivative (amounts to the same thing as path-dependence, here). But as long as we're asking about cross derivative equality, why not go back to scalars themselves?

13.2.1 Scalar Fields

Think of the usual partial derivative of a scalar field ϕ in flat space – we know that $\phi_{,\alpha\beta} = \phi_{,\beta\alpha}$, it doesn't matter if we take the x-derivative or y-derivative first or second (in colloquial terms). Is this true of the covariant derivative? We are interested because in our spaces, partial derivatives do not, in general, lead to tensor behavior.

The first derivative of a scalar is a covariant vector – let $f_{\alpha} = \phi_{,\alpha}$. Fine, but the second derivative is now a covariant derivative acting on f_{α}:

$$f_{\alpha;\beta} = \phi_{,\alpha\beta} - \Gamma^{\sigma}{}_{\alpha\beta}\,\phi_{,\sigma}. \tag{13.7}$$

Then the difference $\phi_{;\alpha\beta} - \phi_{\beta;\alpha}$ is

$$f_{\alpha;\beta} - f_{\beta;\alpha} = -\left(\Gamma^{\sigma}{}_{\alpha\beta} - \Gamma^{\sigma}{}_{\beta\alpha}\right)\,\phi_{,\sigma}. \tag{13.8}$$

Now we know from the connection of the connection to the metric (no pun intended) that $\Gamma^{\sigma}{}_{\alpha\beta} = \Gamma^{\sigma}{}_{\beta\alpha}$, but this is a negotiable point for more general manifolds. If the connection is not symmetric, its antisymmetric part can be associated with the "torsion" of the manifold. For our metrically-connected spaces, the torsion is zero.

We conclude that, for us,

$$\boxed{\phi_{;\alpha\beta} = \phi_{;\beta\alpha},} \tag{13.9}$$

scalars have cross-covariant-derivative equality.

13.2.2 Vector Fields

Given a vector field $f^\alpha(x)$, we know how to form $f^\alpha(x)_{;\beta}$ but what about the second derivatives? If $f^\alpha{}_{,\beta\gamma} = f^\alpha{}_{,\gamma\beta}$, is it the case that $f^\alpha{}_{;\beta\gamma} = f^\alpha{}_{;\gamma\beta}$? It is by no means obvious that it should be, given all the $\Gamma^\alpha{}_{\beta\gamma}$ lying around, and indeed it isn't true in general.

What we want to know is: "By how much is cross-derivative equality violated?" So begins a long calculation, following the pattern for the scalar case, we will calculate explicitly.

Let $h^\alpha{}_\beta \equiv f^\alpha{}_{;\beta}$, then

$$h^\alpha{}_{\beta;\gamma} = h^\alpha{}_{\beta,\gamma} + \Gamma^\alpha{}_{\gamma\sigma}\, h^\sigma{}_\beta - \Gamma^\sigma{}_{\beta\gamma}\, h^\alpha{}_\sigma, \tag{13.10}$$

so that

$$\begin{aligned} f^\alpha{}_{;\beta\gamma} &= \left(f^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\sigma}\, f^\sigma\right)_{,\gamma} + \Gamma^\alpha{}_{\gamma\sigma}\left(f^\sigma{}_{,\beta} + \Gamma^\sigma{}_{\beta\rho}\, f^\rho\right) - \Gamma^\sigma_{\beta\gamma}\left(f^\alpha{}_{,\sigma} + \Gamma^\alpha{}_{\sigma\rho}\, f^\rho\right) \\ f^\alpha{}_{;\gamma\beta} &= \left(f^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\gamma\sigma}\, f^\sigma\right)_{,\beta} + \Gamma^\alpha{}_{\beta\sigma}\left(f^\sigma{}_{,\gamma} + \Gamma^\sigma{}_{\gamma\rho}\, f^\rho\right) - \Gamma^\sigma_{\gamma\beta}\left(f^\alpha{}_{,\sigma} + \Gamma^\alpha{}_{\sigma\rho}\, f^\rho\right). \end{aligned} \tag{13.11}$$

Now noting that $\Gamma^\alpha{}_{\gamma\beta} = \Gamma^\alpha{}_{\beta\gamma}$ and cross-derivative equality for ∂_μ, we can simplify the difference, it is

$$\boxed{\begin{aligned} f^\alpha{}_{;\beta\gamma} - f^\alpha{}_{;\gamma\beta} &= \left(\Gamma^\alpha{}_{\gamma\sigma}\,\Gamma^\sigma{}_{\beta\rho} - \Gamma^\alpha{}_{\beta\sigma}\,\Gamma^\sigma{}_{\gamma\rho} + \Gamma^\alpha{}_{\beta\rho,\gamma} - \Gamma^\alpha{}_{\gamma\rho,\beta}\right)\, f^\rho \\ &\equiv R^\alpha{}_{\rho\gamma\beta}\, f^\rho, \end{aligned}} \tag{13.12}$$

with $R^\alpha{}_{\rho\gamma\beta}$ the "Riemann tensor".

There is a long tradition of definition here. That the above mess in parentheses is relevant enough to warrant its own letter and dangling indices should not be particularly clear to us at this point. I am going to try to show you how one might approach it in another way, and by so doing, give meaning to this object. One thing that should be somewhat surprising, and indicate the deep significance of the Riemann tensor, is that we started with a discussion of a vector f^α, took some derivatives and found that the result depended only linearly on f^α itself – i.e. the Riemann tensor is interesting in that it is independent of f^α – any vector is proportional to the same deviation from cross-derivative equality.

Another impressive aspect of this tensor is its complicated relationship to the metric – if we input the Christoffel connection in terms of the metric and its derivatives, we have terms $\sim g^2$ as well as $\partial\partial g$ terms, a coupled set of nonlinear PDE's.

Well, that's fine, we are defining the Riemann tensor (four indices) to be the extent to which covariant derivatives don't commute, not exactly breathtaking. Let me just write it explicitly without the baggage:

$$\boxed{R^{\alpha}{}_{\rho\gamma\beta} \equiv \left(\Gamma^{\alpha}{}_{\gamma\sigma}\,\Gamma^{\sigma}{}_{\beta\rho} - \Gamma^{\alpha}{}_{\beta\sigma}\,\Gamma^{\sigma}{}_{\gamma\rho} + \Gamma^{\alpha}{}_{\beta\rho,\gamma} - \Gamma^{\alpha}{}_{\gamma\rho,\beta}\right).} \tag{13.13}$$

13.3 Interpretation

With our parallel transport law, we know how to move vectors around from point to point, but what path should we take? In flat space with Cartesian coordinates, the path doesn't matter, moving the vector is simply a matter of adding to its components, and we never even talk about the path along which we move, we just move the thing. In a curved space, though, we can imagine scenarios where the path *does* matter, and so we must be able to quantify the effects of taking different paths. After all, if we make a carbon copy of ourselves standing around with our vector f^{α} at point P, and then we go off, proudly showing off our vector at neighbouring points, it would be a little embarrasing to return and find that our vector was rotated without knowing why.

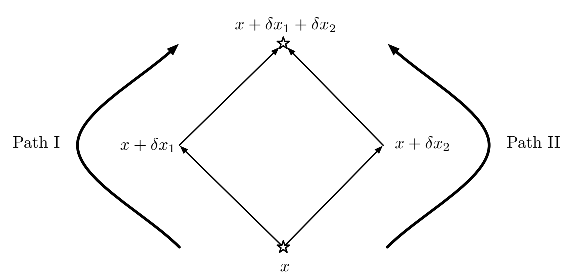

Figure 13.1: The two paths used to propagate a vector from x to $x+\delta x_1+\delta x_2$.

We will construct the difference between a vector moved along two different paths. Referring to Figure 13.1, we take path I from $x \rightarrow x + \delta x_1 \rightarrow x + \delta x_1 + \delta x_2$ and path II from $x \rightarrow x + \delta x_2 \rightarrow x + \delta x_2 + \delta x_1$. Then we can just subtract the two vectors at the end point to find the residual.

At $x + \delta x_1$ along path I, we have, by Taylor's theorem:

$$f^{\alpha}(x + \delta x_1) = f^{\alpha}(x) + \delta x_1^{\gamma}\, f^{\alpha}{}_{,\gamma}(x), \tag{13.14}$$

and we have been careful to parallel transport our vector so that along the path connecting the two points, we have $\delta x^\gamma f^\alpha{}_{;\gamma} = 0$. Because of this, partial derivatives in the above can be replaced via: $f^\alpha{}_{,\gamma}\, \delta x_1^\gamma = -\Gamma^\alpha{}_{\gamma\beta}\, f^\beta$. Putting this in, our parallel transported vector takes on the value

$$f^\alpha(x + \delta x_1) = f^\alpha(x) - \Gamma^\alpha{}_{\beta\gamma}(x)\, f^\beta(x)\, \delta x_1^\gamma \tag{13.15}$$

(the substitution that we made here highlights another aspect of parallel transport – it moves the vector in such a way that the coordinate derivatives cancel the basis derivatives).

Now starting from $x+\delta x_1$, we use Taylor's theorem again to get to $x+\delta x_1+\delta x_2$:

$$\begin{aligned} f^\alpha((x+\delta x_1)+\delta x_2) &= f^\alpha(x+\delta x_1) + f^\alpha{}_{,\gamma}(x+\delta x_1)\, \delta x_2^\gamma \\ &= f^\alpha(x+\delta x_1) - \Gamma^\alpha{}_{\beta\gamma}(x+\delta x_1)\, f^\beta(x+\delta x_1)\, \delta x_2^\gamma \end{aligned} \tag{13.16}$$

We have to expand the Christoffel symbol about the point x (using guess what) and we can also input the result from (13.15) to write the vector at $x + \delta x_1 + \delta x_2$ along path I entirely in terms of elements evaluated at x:

$$\begin{aligned} f^\alpha((x+\delta x_1)+\delta x_2) = &\left(f^\alpha(x) - \Gamma^\alpha{}_{\beta\gamma}(x)\, f^\beta(x)\, \delta x_1^\gamma\right) \\ &- \delta x_2^\gamma \left(\Gamma^\alpha{}_{\beta\gamma}(x) + \delta x_1^\sigma\, \Gamma^\alpha{}_{\beta\gamma,\sigma}(x)\right) \left(f^\beta(x) - \Gamma^\beta_{\gamma\sigma}(x)\, f^\sigma(x)\right). \end{aligned} \tag{13.17}$$

Finally, we can expand to order δx^2 (everything evaluated at x)

$$f^\alpha((x+\delta x_1)+\delta x_2) = f^\alpha - \Gamma^\alpha{}_{\beta\gamma}\, f^\beta\, (\delta x_1^\gamma + \delta x_2^\gamma) - \delta x_1^\gamma\, \delta x_2^\rho\, f^\sigma \left(\Gamma^\alpha{}_{\beta\gamma}\, \Gamma^\beta{}_{\sigma\rho} - \Gamma^\alpha{}_{\sigma\gamma,\rho}\right) \tag{13.18}$$

(plus terms of order δx^3). Taking $\delta x_1 \leftrightarrow \delta x_2$, we can write the equivalent expression for path II:

$$f^\alpha((x+\delta x_2)+\delta x_1) = f^\alpha - \Gamma^\alpha{}_{\beta\gamma}\, f^\beta\, (\delta x_1^\gamma + \delta x_2^\gamma) - \delta x_2^\gamma\, \delta x_1^\rho\, f^\sigma \left(\Gamma^\alpha{}_{\beta\gamma}\, \Gamma^\beta{}_{\sigma\rho} - \Gamma^\alpha{}_{\sigma\gamma,\rho}\right). \tag{13.19}$$

The difference between these two gives us the path dependence:

$$\boxed{\begin{aligned} f^\alpha_{II} - f^\alpha_I &= -\delta x_1^\rho\, \delta x_2^\gamma\, f^\sigma \left(\Gamma^\alpha{}_{\beta\gamma}\, \Gamma^\beta{}_{\sigma\rho} - \Gamma^\alpha{}_{\beta\rho}\, \Gamma^\beta{}_{\sigma\gamma} - \Gamma^\alpha{}_{\sigma\gamma,\rho} + \Gamma^\alpha{}_{\sigma\rho,\gamma}\right) \\ &= \delta x_1^\rho\, \delta x_2^\gamma\, f^\sigma\, R^\alpha{}_{\sigma\rho\gamma}. \end{aligned}} \tag{13.20}$$

We see that the difference between the vector f^α at the point $x + \delta x_1 + \delta x_2$ as calculated along the two different paths is proportional to the Riemann tensor.

How does that relate to this question of coordinates vs. geometry? Again, in flat space, path independence is built in, that means, from the above, that $R^\alpha{}_{\sigma\rho\gamma}$ must be zero throughout the space. Then we can immediately tell whether a space is flat. This connection can be tightened – when we talk about flat space, the implicit definition is that we can find coordinates such that the metric has constants along the diagonal. Such a metric certainly has zero for all entries of the Riemann tensor. One can also show that path-independence implies that coordinates can be chosen such that the metric is constant – that's the missing piece of the puzzle here, that the vanishing of $R^\alpha{}_{\beta\gamma\delta}$ implies the existence of a coordinate transformation that puts the metric in flat form:

$$g_{\mu\nu} \doteq \begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}. \tag{13.21}$$

We can show that this is true by constructing the coordinates explicitly[3]. The theorem is: "A space-time is flat if and only if its Riemann tensor vanishes." One direction is trivial (flat space has zero Riemann tensor – obvious since its connection vanishes everywhere), now we go the other direction.

Suppose we have $R_{\alpha\beta\gamma\delta} = 0$ – then parallel transport of vectors is path-independent, and we can uniquely define a vector field $f_\alpha(x)$ at every point simply by specifying its value at a single point – the PDE we must solve is

$$f_{\alpha,\gamma} - \Gamma^\sigma_{\alpha\gamma} f_\sigma = 0. \tag{13.22}$$

Take f_α to be the gradient of a scalar ϕ – then

$$\phi_{,\alpha\gamma} = \Gamma^\sigma_{\alpha\gamma} \phi_{,\sigma} \tag{13.23}$$

and we know this PDE is also path-independent since the Christoffel connection is symmetric. Now choose four independent scalar solutions to the

[3] This argument comes from the Dirac lecture notes, and his treatment is concise and simple – so much so that I cannot resist repeating it here. An expanded version can be found in D'Inverno.

above, ϕ^ρ, and use these to define a new set of coordinates $\bar{x}^\rho = \phi^\rho$, so these new coordinates themselves are solutions to

$$\bar{x}^\rho{}_{,\mu\nu} = \Gamma^\sigma{}_{\mu\nu}\, \bar{x}^\rho{}_{,\sigma} \qquad \bar{x}^\rho{}_{,\sigma} \equiv \frac{\partial \bar{x}^\rho}{\partial x^\sigma} \tag{13.24}$$

Then the metric transforms in the usual way, and we can write the original metric in terms of the new one via:

$$g_{\alpha\beta} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta}\,\bar{g}_{\mu\nu}. \tag{13.25}$$

Our goal is to show that the new metric has zero partial derivatives, hence must be constant and therefore flat – if we take the x^γ derivative of both sides of the metric transformation, we have

$$g_{\alpha\beta,\gamma} = \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\gamma}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} + \bar{g}_{\mu\nu}\left(\bar{x}^\mu{}_{,\alpha\gamma}\,\bar{x}^\nu{}_{,\beta} + \bar{x}^\mu{}_{,\alpha}\,\bar{x}^\nu{}_{,\beta\gamma}\right), \tag{13.26}$$

and using the definition of the coordinates (13.24), we have

$$\begin{aligned} g_{\alpha\beta,\gamma} &= \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\gamma}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} + \bar{g}_{\mu\nu}\left[\Gamma^\sigma{}_{\alpha\gamma}\,\bar{x}^\mu{}_{,\sigma}\,\bar{x}^\nu{}_{,\beta} + \Gamma^\sigma{}_{\beta\gamma}\,\bar{x}^\mu{}_{,\alpha}\,\bar{x}^\nu{}_{,\sigma}\right] \\ &= \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\gamma}\frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} + \Gamma^\sigma{}_{\alpha\gamma}\,g_{\sigma\beta} + \Gamma^\sigma_{\beta\gamma}\,g_{\alpha\sigma} \\ &= \frac{\partial \bar{g}_{\mu\nu}}{\partial x^\gamma}\,\bar{x}^\mu{}_{,\alpha}\,\bar{x}^\nu{}_{,\beta} + g_{\alpha\beta,\gamma} \end{aligned} \tag{13.27}$$

where the second line follows from (13.25), and the final line comes from the relationship of the connection to derivatives of the metric. We have, finally

$$\frac{\partial \bar{g}_{\mu\nu}}{\partial x^\gamma}\,\bar{x}^\mu{}_{,\alpha}\,\bar{x}^\nu{}_{,\beta} = \bar{g}_{\mu\nu,\rho}\frac{\partial \bar{x}^\rho}{\partial x^\gamma}\,\bar{x}^\mu{}_{,\alpha}\,\bar{x}^\nu{}_{,\beta} = 0 \tag{13.28}$$

and assuming the transformation is invertible, this tells us that $\bar{g}_{\mu\nu,\rho} \equiv \frac{\partial \bar{g}_{\mu\nu}}{\partial \bar{x}^\rho} = 0$, so the metric is constant. This program is carried out for polar coordinates below.

13.3.1 Example – Polar Coordinates

Suppose we were given metric and coordinates in two (spatial) dimensions:

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} \qquad x^\mu \doteq \begin{pmatrix} s \\ \phi \end{pmatrix}, \tag{13.29}$$

then the non-zero connection coefficients are:

$$\Gamma^s{}_{\phi\phi} = -s \qquad \Gamma^\phi{}_{s\phi} = \Gamma^\phi{}_{\phi s} = \frac{1}{s}, \tag{13.30}$$

as we have computed before.

Our recipe is to solve the PDE (13.23) (replacing the scalar ϕ with ψ for obvious reasons) – written out in matrix form, this provides three independent equations:

$$\begin{pmatrix} \frac{\partial^2\psi}{\partial s^2} & \frac{\partial^2\psi}{\partial\phi\partial s} \\ \frac{\partial^2\psi}{\partial\phi\partial s} & \frac{\partial^2\psi}{\partial\phi^2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{s}\frac{\partial\psi}{\partial\phi} \\ \frac{1}{s}\frac{\partial\psi}{\partial\phi} & -s\frac{\partial\psi}{\partial s} \end{pmatrix}. \tag{13.31}$$

From $\frac{\partial^2\psi}{\partial s^2} = 0$, we learn that $\psi = a(\phi) + s b(\phi)$, and from the lower right-hand equation, we have

$$s\,(b'' + b) = -a'' \tag{13.32}$$

which, since both sides depend only on ϕ tells us that $a(\phi) = a_0\,\phi + a_1$ for constants a_0 and a_1, and $b(\phi) = b_0\,\cos\phi + b_1\,\sin\phi$. The final scalar solution reads, once a_0 is set to zero to satisfy the off-diagonal equation,

$$\psi = s\,(b_0\,\cos\phi + b_1\,\sin\phi). \tag{13.33}$$

Next, we take two of these solutions to define the new coordinates – how about $\bar{x}^1 = s\,\cos\phi$ and $\bar{x}^2 = s\,\sin\phi$? This is precisely the usual two-dimensional Cartesian coordinates for flat space. If we calculate the transformation of the metric:

$$\bar{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial\bar{x}^\mu}\,\frac{\partial x^\beta}{\partial\bar{x}^\nu}\,g_{\alpha\beta} \tag{13.34}$$

we learn that

$$\bar{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.35}$$

The moral: If someone hands you a metric in whatever coordinates they want, you could try to exhibit a coordinate transformation that puts the metric in constant diagonal form, or you could just calculate the Riemann tensor for the space and verify that it is or is not zero.