Continuity Equations and the Energy-Momentum Tensor

Lecture 18

Physics 411
Classical Mechanics II

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We have finished the definition of Lagrange density for a generic space-time described by a metric $g_{\mu\nu}$. Now we must carry out the actual change-of-variables and see how the action changes formally. The fact that the action cannot change under a change-of-variables will lead to some constraints on the field solutions (and their derivatives). This is an example of Noether's theorem, and we will recover a number of familiar results.

Keep in mind that all we have discussed so far are free (scalar) fields, that is, functions that are self-consistent but source-less. We have not yet described the connection between sources and fields, nor the physical effect of fields on test particles. Both field sources and field "forces" can be put into our current Lagrange framework, but before we do that, we must find the explicit manifestation of coordinate invariance for the fields. For this, we do not need more than the free field action.

While the following refers to general field actions (scalar, vector, and higher), it is useful to continue our connection to scalar fields. In addition to developing the energy-momentum tensor and its set of four immediate continuity equations, we will also look at a Hamiltonian-ization of the Lagrange program, an equivalent formulation of the field equations and one which will help with some interpretative issues that show up in the energy-momentum tensor itself.

18.1 Coordinates and Conservation

Returning to our discussion of the coordinate invariance of the action, the idea is to make a formal, infinitesimal coordinate transformation, see what changes (to first order) that induces in the action $S \rightarrow S + \delta S$ and require that $\delta S = 0$, enforcing the overall isometric nature of S. We begin with the action

$$S = \int \mathcal{L}(\phi, \phi_{,\mu}, g_{\mu\nu})\, d\tau \tag{18.1}$$

where $\mathcal{L} = \sqrt{-g}\,\bar{\mathcal{L}}(\phi, \phi_{,\mu})$ is the Lagrange density, and we indicate the metric dependence in $\mathcal{L}$. Now we introduce an infinitesimal coordinate transformation:

$$x'^{\mu} = x^{\mu} + \eta^{\mu} \tag{18.2}$$

with η^{μ} a small, but general perturbation $\eta^{\mu} = \eta^{\mu}(x)$. The coordinate transformation generates a perturbation to both the field $\phi(x)$ and the metric $g_{\mu\nu}(x)$ – let's write these as $\phi'(x') = \phi(x) + \delta\phi(x)$ and $g'_{\mu\nu}(x') = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$. We can connect these explicitly to the perturbation $\eta^{\mu}(x)$ by Taylor expansion – take $\phi(x')$:

$$\phi'(x') = \phi(x') = \phi(x + \eta) = \phi(x) + \frac{\partial\phi}{\partial x^{\mu}}\,\eta^{\mu} + O(\eta^2) \tag{18.3}$$

so we would call $\delta\phi = \frac{\partial\phi}{\partial x^{\mu}}\,\eta^{\mu}$. We'll return to the analagous procedure for $g'_{\mu\nu}(x')$ in a moment.

Viewing the Lagrange density as a function of ϕ, $\phi_{,\mu}$ and $g_{\mu\nu}$ we have

$$\mathcal{L}(\phi+\delta\phi, \phi_{,\mu}+\delta\phi_{,\mu}, g_{\mu\nu}+\delta g_{\mu\nu}) \approx \mathcal{L}(\phi, \phi_{,\mu}, g_{\mu\nu}) + \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\,\delta\phi_{,\mu} + \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\,\delta g_{\mu\nu}, \tag{18.4}$$

and it is pretty clear what will happen to the action when we put this in and integrate by parts – we'll get:

$$S'[\phi + \delta\phi] = S[\phi] + \underbrace{\int \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} \right) \right) \delta\phi\, d\tau}_{=0} + \underbrace{\int \frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\,\delta g_{\mu\nu} d\tau}_{\equiv\delta S}. \tag{18.5}$$

Notice that the first perturbative term above vanishes by the field equations – indeed, this first term is just the result of an induced variation of ϕ (the field equation for ϕ comes from arbitrary variation, we are just picking a particular variation $\delta\phi$ associated with coordinate transformations, so the field equations still hold). The second term, labelled δS must be zero for S to remain unchanged. Now we need to connect the change in the metric to the change in coordinates. It is tempting to simply set $\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = 0$, but this is overly restrictive ($\delta g_{\mu\nu}$ here refers to a *specific* type of variation for the metric, not a generic one).

Consider the general transformation of a second rank covariant tensor

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\,\frac{\partial x^\beta}{\partial x'^\nu}\,g_{\alpha\beta}(x). \tag{18.6}$$

If we re-write the transformation in terms of the original variables (so that we can recover S in its original form), then

$$g'_{\mu\nu}(x) = g_{\mu\nu} + \left(-\eta^\beta{}_{,\nu}\,g_{\mu\beta} - \eta^\alpha{}_{,\mu}\,g_{\alpha\nu} - g_{\mu\nu,\sigma}\,\eta^\sigma\right) \tag{18.7}$$

with both sides functions of x now.

A moral point: we rarely (if ever) leave partial derivatives alone – a comma (as in $g_{\mu\nu,\sigma}$) is, as we have learned, not a tensor operation. The whole point of introducing covariant differentiation was to generate tensor character, so it is always a good idea to use it. We are in luck, the metric's covariant derivative vanishes (by definition, effectively, but we saw this a while ago), so that:

$$g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - \Gamma^\alpha{}_{\mu\sigma}\,g_{\alpha\nu} - \Gamma^\alpha{}_{\sigma\nu}\,g_{\mu\alpha} = 0. \tag{18.8}$$

The partial derivative on $\eta^\beta{}_{,\nu}$ can also be replaced via

$$\eta^\beta{}_{;\nu} = \eta^\beta{}_{,\nu} + \Gamma^\beta{}_{\sigma\nu}\,\eta^\sigma, \tag{18.9}$$

and similarly for the $\eta^\alpha{}_{,\mu}$. Until, finally, we have

$$g'_{\mu\nu}(x) = g_{\mu\nu} - \eta_{\mu;\nu} - \eta_{\nu;\mu}. \tag{18.10}$$

Evidently, the perturbation is $\delta g_{\mu\nu} = -(\eta_{\mu;\nu} + \eta_{\nu;\mu})$ and we are, finally, ready to return to δS.

Referring to (18.5), we can define a new density built from a tensor $T^{\mu\nu}$

$$\frac{1}{2}\sqrt{-g}\,T^{\mu\nu} \equiv -\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \tag{18.11}$$

which is symmetric, so that

$$\begin{aligned}\delta S &= -\frac{1}{2}\int d\tau\,\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu} = +\frac{1}{2}\int d\tau\,\sqrt{-g}\,T^{\mu\nu}\,(\eta_{\mu;\nu} + \eta_{\nu;\mu}) \\ &= \int d\tau\,\sqrt{-g}\,T^{\mu\nu}\,\eta_{\mu;\nu}.\end{aligned} \tag{18.12}$$

We have used the symmetric property of $T^{\mu\nu}$ to simplify (notice how the strange factor of $\frac{1}{2}$ and the minus sign from (18.11) have played a role). Now we can break the above into a total derivative, which will turn into a surface integral that vanishes (by assumption, on the boundary of the integration, η^μ goes away) and an additional term:

$$\int d\tau \sqrt{-g}\, T^{\mu\nu}\, \eta_{\mu;\nu} = \underbrace{\int d\tau \left(\sqrt{-g}\, T^{\mu\nu}\, \eta_\mu\right)_{;\nu}}_{=0} - \int d\tau \sqrt{-g}\, T^{\mu\nu}{}_{;\nu}\, \eta_\mu. \quad (18.13)$$

For arbitrary $\eta_\mu(x)$, then, we have

$$T^{\mu\nu}{}_{;\nu} = 0. \quad (18.14)$$

18.2 The Stress-Energy Tensor

Think of a four-divergence in Minkowski space – an expression of the form $A^\mu_{;\mu} = 0$ reduces to (in flat space, ; →, and we take Minkowski in its usual Cartesian form)

$$\frac{1}{v}\frac{\partial A^0}{\partial t} + \nabla \cdot \mathbf{A} = 0 \longrightarrow \frac{\partial A^0}{\partial t} = -v\, \nabla \cdot \mathbf{A} \quad (18.15)$$

with $\mathbf{A}$ the spatial components of the four-vector A^μ. This is a continuity statement – in generic language, we say that the zero component is the conserved "charge", and the vector portion the "current", terms coming from one of the most common examples: charge conservation where $J^\mu \doteq (c\,\rho, J_x, J_y, J_z) = (c\,\rho, \mathbf{J})$, and then

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} \quad (18.16)$$

(with $v = c$ for E&M). The Lorentz gauge condition can also be expressed this way, with $A^\mu \doteq (\phi/c, A_x, A_y, A_z)^T$, leading to

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{A}. \quad (18.17)$$

For a second rank tensor in this setting, the statement (18.14) becomes four equations

$$\begin{aligned} 0 &= \frac{1}{v}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0j}}{\partial x^j} \qquad \text{j = 1,2,3} \\ 0 &= \frac{1}{v}\frac{\partial T^{j0}}{\partial t} + \frac{\partial T^{jk}}{\partial x^k}. \end{aligned} \quad (18.18)$$

If we integrate over an arbitrary spatial volume, and use the usual form of Gauss's law, we can interpret these four equations as continuity equations as well

$$\begin{aligned}\frac{\partial}{\partial t}\int \frac{1}{v}\,T^{00}\,d\tau &= -\oint T^{0j}\,da_j\\ \frac{\partial}{\partial t}\int \frac{1}{v}\,T^{0j}\,d\tau &= -\oint T^{jk}\,da_k,\end{aligned} \tag{18.19}$$

with the obvious identification of a scalar and three-vector in flat space.

In order to understand the actual physics of this $T^{\mu\nu}$ tensor, we will find the explicit form in terms of the Lagrange density and see what this implies about our scalar field.

18.2.1 The Tensor $T^{\mu\nu}$ in General

From its definition:

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\,\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}, \tag{18.20}$$

we need the derivative of the density $\mathcal{L}$ in terms of $g_{\mu\nu}$. A typical density, like our scalar field, will depend on $g_{\mu\nu}$ through g, the determinant, and potentially, a metric that is used to contract terms like $\phi_{,\mu}\,g^{\mu\nu}\,\phi_{,\nu}$. Let's assume the form is

$$\mathcal{L}(\phi, \phi_{,\mu}, g_{\mu\nu}) = \sqrt{-g}\,\bar{\mathcal{L}}(\phi, \phi_{,\mu}, g_{\mu\nu}) \tag{18.21}$$

and note that $\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}$, so

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\bar{\mathcal{L}} + \sqrt{-g}\,\frac{\partial \bar{\mathcal{L}}}{\partial g_{\mu\nu}}. \tag{18.22}$$

The tensor of interest is

$$T^{\mu\nu} = -\left(g^{\mu\nu}\,\bar{\mathcal{L}} + 2\,\frac{\partial \bar{\mathcal{L}}}{\partial g_{\mu\nu}}\right). \tag{18.23}$$

18.2.2 Free Scalar Field (Minkowski)

For the free scalar field, the Lagrange scalar is $\bar{\mathcal{L}} = \frac{1}{2}\,\phi_{,\alpha}\,g^{\alpha\beta}\,\phi_{,\beta}$. If we input this Lagrange scalar in (18.23), we need to evaluate the term[1]

$$\frac{\partial \bar{\mathcal{L}}}{\partial g_{\mu\nu}} = -\frac{1}{2}\,\phi_{,\alpha}\,\phi_{,\beta}\,g^{\alpha\mu}\,g^{\beta\nu}, \tag{18.24}$$

and the $T^{\mu\nu}$ tensor is

$$T^{\mu\nu} = \phi_{,\alpha}\,\phi_{,\beta}\,g^{\alpha\mu}\,g^{\beta\nu} - \frac{1}{2}\,g^{\mu\nu}\,\phi_{,\alpha}\,g^{\alpha\beta}\,\phi_{,\beta}. \tag{18.25}$$

Again, we have the question of interpretation here – referring to our original discrete Lagrangian, from whence all of this came, we can transform to a Hamiltonian, and this will give us an expression (upon taking the continuum limit) for what might reasonably be called the energy density of the field. The procedure is motivated by its classical analogue – where the temporal derivative is privileged – this seems strange in our current homogeneous treatment, but lends itself to an obvious energetic interpretation.

In the particle case, the canonical momentum associated with the motion of an individual portion of the string (for example) would be $\frac{\partial L}{\partial \dot{\phi}(\bar{x}_j,t)} = \mu\,\dot{\phi} = \pi$ – suggesting that we take, as the canonical momentum for the field ϕ: $\frac{\partial \bar{\mathcal{L}}}{\partial \dot{\phi}} \equiv \pi$. Starting from

$$\bar{\mathcal{L}} = \frac{1}{2}\,\phi_{,\mu}\,g^{\mu\nu}\,\phi_{,\nu} = \frac{1}{2}\left(-\frac{1}{v^2}\,\dot{\phi}^2 + \phi'^2\right) \tag{18.26}$$

we identify $\pi = -\frac{1}{v^2}\,\dot{\phi}$, and then

$$\begin{aligned}\bar{\mathcal{H}} &= \dot{\phi}\,\pi - \bar{\mathcal{L}} = -\frac{1}{v^2}\,\dot{\phi}^2 - \left(-\frac{1}{2\,v^2}\,\dot{\phi}^2 + \frac{1}{2}\,\phi'^2\right)\\ &= -\frac{1}{2}\left(\frac{1}{v^2}\,\dot{\phi}^2 + \phi'^2\right) \equiv -\mathcal{E},\end{aligned} \tag{18.27}$$

where we define the energy density $\mathcal{E}$ to be the negative of $\bar{\mathcal{H}}$ (this is just a matter of the metric signature, nothing deep). Compare this with T^{00} from (18.25) – that component is

$$\begin{aligned}T^{00} &= \phi_{,0}\,\phi_{,0} + \frac{1}{2}\left(-\phi_{,0}^2 + \phi_{,1}^2\right) = \frac{1}{2}\left(\frac{1}{v^2}\,\dot{\phi}^2 + \phi'^2\right)\\ &= \mathcal{E},\end{aligned} \tag{18.28}$$

[1] Using $\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -g^{\alpha\mu}\,g^{\beta\nu}$

so that the zero-zero (pure temporal) component of $T^{\mu\nu}$ is naturally identified with the energy density of the system.

If we introduce our overall factor (multiply by $\mu\, v^2$ in $\bar{\mathcal{L}}$) to make contact with real longitudinal oscillations, then the energy density is (with units)

$$\mathcal{E} = \frac{1}{2}\,\mu\,\dot{\phi}^2 + \frac{1}{2}\,\mu\,v^2\,\phi'^2 \tag{18.29}$$

which is pretty clearly the kinetic and potential energy per unit length (think of what we would have gotten out of the Hamiltonian for balls and springs).

Now we want to find the natural momentum density here, and we can work directly from the energy for a short segment of string $E = \int_a^b \mathcal{E}\,dx$ – as time goes on, energy will flow into and out of this segment, and we can calculate the temporal dependence of that flow via:

$$\begin{aligned} E(t+dt) = E(t) + \frac{dE}{dt}\,dt &= \int \left[\frac{1}{2}\,\mu\,(\dot{\phi}(t+dt))^2 + \frac{1}{2}\,\mu\,v^2\,(\phi'(t+dt))^2\right] dx \\ &= \int \left[\frac{1}{2}\,\mu\,\dot{\phi}(t)^2 + \frac{1}{2}\,\mu\,v^2\,\phi'(t)^2 + dt\left(\mu\,\ddot{\phi}\,\dot{\phi} + \mu\,v^2\,\phi'\,\dot{\phi}'\right)\right]\,dx, \end{aligned} \tag{18.30}$$

and using the field equation: $\ddot{\phi} = v^2\,\phi''$, we can write the derivative as:

$$\frac{dE}{dt} = \int \frac{\partial}{\partial x}\left(\mu\,v^2\,\dot{\phi}\,\phi'\right)\,dx \tag{18.31}$$

which, as a total derivative, can be evaluated at the endpoints as usual. In terms of the local statement, we have:

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{\partial}{\partial x}\left(-\mu\,v^2\,\dot{\phi}\,\phi'\right). \tag{18.32}$$

This is supposed to be related to the T^{0x} element of the stress tensor – we expect, from (18.18), that: $T^{0x} = \mu\,v\,\dot{\phi}\,\phi'$ – but that is precisely what we get from (18.25). In addition, we can interpret this component in terms of momentum density – for a patch of field between x and $x + dx$, we have mass displaced from x into the interval: $\phi(x)$, and mass displaced out of the interval on the right: $\phi(x + dx)$ – then the momentum in this interval is

$$\mathfrak{p}dx = \mu\,\dot{\phi}\,(\phi(x) - \phi(x+dx))\,dx \sim \mu\,\dot{\phi}\,\phi'\,dx \tag{18.33}$$

and this is related to T^{0x} by one factor of v: $\mathfrak{p} = \frac{1}{v}\,T^{0x}$.