Scalar Fields and Gauge

Lecture 23

Physics 411
Classical Mechanics II

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We will discuss the use of multiple fields to expand our notion of symmetries and conservation. Using a natural "current" that comes from complex massive scalar field theory, we have a candidate source for coupling to E&M. We go through the usual procedure of minimal coupling, followed by a process of consistency that we will use to introduce the idea of local gauge invariance. The approach can be thought of in terms of augmenting a Lagrangian until a consistent theory is obtained, and displays the pattern of guess-check-reguess-recheck that in this case ends after one iteration.

23.1 Two Scalar Fields

Two noninteracting, massive scalar fields can be developed from our current considerations – the simplest representation would be

$$\bar{\mathcal{L}} = \left(\frac{1}{2}\,\psi_{,\mu}\,\psi_{,}{}^{\mu} - \frac{1}{2}\,m^2\,\psi^2\right) + \left(\frac{1}{2}\,\eta_{,\mu}\,\eta_{,}{}^{\mu} - \frac{1}{2}\,m^2\,\eta^2\right), \tag{23.1}$$

where we've given both fields the same mass. As a sum of two independent Lagrangians, it is easy to see that the variation of ψ and η do not talk to each other, so we will get a pair of massive Klein-Gordon fields.

Given that there are two fields, we "immediately" think of the real and imaginary parts of a complex number, and define the *independent* fields ϕ and ϕ^* by taking $\phi \equiv \psi + i\,\eta$, $\phi^* = \psi - i\,\eta$. Then in these variables,

$$\bar{\mathcal{L}} = \phi^*_{,\mu}\,g^{\mu\nu}\,\phi_{,\nu} - m^2\,\phi^*\,\phi \tag{23.2}$$

and to obtain the field equations, we need to vary w.r.t ϕ and ϕ^* independently (it's a two-field theory, after all). The two field equations are symmetric

$$\begin{aligned} 0 &= -\Box^2\phi - m^2\,\phi \\ 0 &= -\Box^2\phi^* - m^2\,\phi^*. \end{aligned} \tag{23.3}$$

It is pretty clear from the action that our variation, w.r.t $\delta\phi$ and $\delta\phi^*$, which constrains the fields themselves, says nothing at all about invariance of the Lagrangian under variations that *mix* the fields. For example, if we take $\phi \rightarrow e^{i\,\alpha}\,\phi$, then $\phi^* \rightarrow e^{-i\,\alpha}\,\phi^*$ doesn't change the Lagrangian at all, yet corresponds to a coupling in the variation between the two fields ϕ and ϕ^* (this is easy to see in the real formulation (23.1) above). That is, we are linking the two fields together through the phase factor α, a constant. We know that any value for α leaves the action unchanged, and so it is reasonable to ask what this implies about conservation (a Noetherian question).

But given that we cannot directly vary w.r.t α, we need a new way to think about this type of symmetry. Suppose we take the infinitesimal form of $e^{i\alpha} \approx (1 + i\,\alpha)$ – then we are looking at the transformation

$$\begin{aligned} \phi &\longrightarrow \phi\,(1 + i\,\alpha) \\ \phi^* &\longrightarrow \phi^*\,(1 - i\,\alpha), \end{aligned} \tag{23.4}$$

so we can write the perturbed Lagrangian:

$$\bar{\mathcal{L}}(\phi + \epsilon_1\,\phi, \phi^* + \epsilon_2\,\phi^*, (\phi + \epsilon_1\,\phi)_{,\mu}, (\phi^* + \epsilon_2\,\phi^*)_{,\mu}) \tag{23.5}$$

with $\epsilon_1 \equiv i\,\alpha$, $\epsilon_2 \equiv -i\,\alpha$ and the Taylor expansion gives

$$0 = \delta\bar{\mathcal{L}} \approx \bar{\mathcal{L}} + \frac{\partial\bar{\mathcal{L}}}{\partial\phi}\,\epsilon_1\,\phi + \frac{\partial\bar{\mathcal{L}}}{\partial\phi^*}\,\epsilon_2\,\phi^* + \frac{\partial\bar{\mathcal{L}}}{\partial\phi_{,\mu}}\,\epsilon_1\,\phi_{,\mu} + \frac{\partial\bar{\mathcal{L}}}{\partial\phi^*_{,\mu}}\,\epsilon_2\,\phi^*_{,\mu}. \tag{23.6}$$

We are using the fact that for ϵ_1 and ϵ_2 defined according to $e^{i\,\alpha}$, we know that the Lagrangian is invariant, so $\delta\bar{\mathcal{L}} = 0$ automatically. From the field equations

$$\begin{aligned} 0 &= \partial_\mu\left(\frac{\partial\bar{\mathcal{L}}}{\partial\phi_{,\mu}}\right) - \frac{\partial\bar{\mathcal{L}}}{\partial\phi} \\ 0 &= \partial_\mu\left(\frac{\partial\bar{\mathcal{L}}}{\partial\phi^*_{,\mu}}\right) - \frac{\partial\bar{\mathcal{L}}}{\partial\phi^*}, \end{aligned} \tag{23.7}$$

we can replace the first two terms in $\delta\bar{\mathcal{L}}$:

$$\begin{aligned} 0 &= \partial_\mu\left(\frac{\partial\bar{\mathcal{L}}}{\partial\phi_{,\mu}}\right)\,\epsilon_1\,\phi + \partial_\mu\left(\frac{\partial\bar{\mathcal{L}}}{\partial\phi^*_{,\mu}}\right)\,\epsilon_2\phi^* + \frac{\partial\bar{\mathcal{L}}}{\partial\phi_{,\mu}}\,\epsilon_1\,\phi_{,\mu} + \frac{\partial\bar{\mathcal{L}}}{\partial\phi^*_{,\mu}}\,\epsilon_2\,\phi^*_{,\mu} \\ &= \partial_\mu \underbrace{\left(\frac{\partial\bar{\mathcal{L}}}{\partial\phi_{,\mu}}\,\epsilon_1\,\phi + \frac{\partial\bar{\mathcal{L}}}{\partial\phi^*_{,\mu}}\,\epsilon_2\,\phi^*\right)}_{\equiv j^\mu}. \end{aligned} \tag{23.8}$$

Explicitly, this global (constant α) phase transformation leads to the conserved "current" (so named because $\partial_\mu j^\mu = 0$ just as with the physical current in E&M):

$$j^\mu = i\,\alpha\left(\left(\partial^\mu \phi^*\right)\phi - \left(\partial^\mu \phi\right)\phi^*\right). \tag{23.9}$$

Again, because of the association with E&M four-currents, the zero component is sometimes called the conserved "charge", and the spatial components the "current". Integrating the conservation statement $\partial_\mu j^\mu = 0$ gives the usual:

$$\frac{1}{c}\frac{d}{dt}\int j^0\,d\tau = -\oint \mathbf{j}\cdot d\mathbf{a}, \tag{23.10}$$

using the Minkowski metric (then $d\tau = dxdydz$ is the spatial three-volume).

23.2 Why We Care

Okay, so that's all very nice, but what does it do for us? Thinking about E&M, we learned last time that any four-vector j^μ that is supposed to provide a source for the electric and magnetic fields must be conserved, $\partial_\mu j^\mu = 0$. That's always true for the physical currents that we use in E&M, swarms of charges, for example. But it makes more sense, in a way, to try to couple the vector field theory for A_μ *to other field theories*. In other words, we take two free field theories, E&M and a complex scalar field, and combine them. In order to do this, we need to have in mind a conserved four-current, and we now know that complex scalar fields have one built-in.

Consider the full action (dispensing with the units for now)

$$S = \int d\tau\,\sqrt{-g}\left(\left(F^{\mu\nu}\left(A_{\nu,\mu} - A_{\mu,\nu}\right) - \frac{1}{2}F^2\right) + \left(\phi^*_{,\mu}\,g^{\mu\nu}\,\phi_{,\nu} - m^2\,\phi\,\phi^*\right) + \alpha\,\tilde{j}^\mu\,A_\mu\right), \tag{23.11}$$

where α is the coupling strength, and $\tilde{j}^\mu \equiv i((\partial^\mu\phi^*)\,\phi - (\partial^\mu\phi)\,\phi^*)$. This is straightforward, but there is a potential problem – we know that $\partial_\mu j^\mu = 0$ for the free fields ϕ and ϕ^*, but by introducing the coupling, it is not clear that it *remains* conserved. This is a persistent issue in classical field theory – we make the simplest possible theory, but we have not yet established that this simple theory is consistent with itself. We see that variation w.r.t. A_μ will couple the scalar fields to E&M, but now there will be A_μ terms in the ϕ and ϕ^* field equations because $\tilde{j}^\mu$ sits next to the four-potential. So we need to check that the whole theory is consistent. That involves finding the field equations and showing, explicitly, that in this expanded setting, $\partial_\mu \tilde{j}^\mu = 0$ (it does not).

The field equations that come from varying A_μ, $F^{\mu\nu}$, ϕ and ϕ^* are:

$$
\begin{aligned}
F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu} \\
F^{\mu\nu}{}_{,\nu} &= \alpha \, \tilde{j}^\mu \\
-\partial^\mu \partial_\mu \, \phi - m^2 \, \phi - \alpha \, i \, \partial^\mu \, (\phi A_\mu) - \alpha \, i \, \partial^\mu \, \phi \, A_\mu &= 0 \qquad \text{and c.c.}
\end{aligned}
\tag{23.12}
$$

Again, by symmetry, we see the need for the conservation of $\tilde{j}^\mu$ – but

$$
\begin{aligned}
\partial_\mu \tilde{j}^\mu &= i \, (\phi \, \partial^\mu \partial_\mu \, \phi^* + \partial^\mu \, \phi^* \, \partial_\mu \, \phi - \phi^* \, \partial^\mu \partial_\mu \, \phi - \partial^\mu \, \phi \, \partial_\mu \, \phi^*) \\
&= i \, (\phi \, \partial^\mu \partial_\mu \, \phi^* - \phi^* \, \partial^\mu \partial_\mu \, \phi)
\end{aligned}
\tag{23.13}
$$

which, using the field equations, becomes

$$
\partial_\mu \tilde{j}^\mu = 2 \, \alpha \, \partial_\mu \, (\phi \, \phi^* \, A^\mu) \neq 0. \tag{23.14}
$$

Aha! Our theory is inconsistent – and now we have the task of fixing it. It is pretty clear how to do this – we basically want to introduce a term that will kill off the above, a term in the Lagrangian whose variation looks like $2 \, \alpha \, \phi \, \phi^* \, A^\mu$. Instead, we will impose local gauge invariance.

23.3 Local Gauge Invariance

We know that the electromagnetic field is unchanged under $A_\mu \rightarrow A_\mu + \psi_{,\mu}$, and that there is an internal phase invariance for the free-scalar fields $\phi \rightarrow e^{i\,\alpha} \, \phi$. How can we combine these two ideas? It is not even particularly clear that we *should* combine them. Why should the coupled system exhibit the same symmetries as the individual ones? As it turns out, this is a fundamental (and new, from our point of view) guiding principle for generating "good" field theories – that somehow, merging two field theories should have as much (or more) gauge structure than the free theories did.

We have basically one option available to us if we want to combine the gauge transformations for A_μ and ϕ – because ψ is an arbitrary function of x, we cannot obtain a relation between ψ and α that holds everywhere. Solution: Make α a function of position, and indeed, let's set it equal to ψ. Now the gauge function $\psi(x)$ is itself a field, and it is clear from the derivatives in the complex scalar Lagrangian that $\phi \rightarrow \phi' = \phi \, e^{i\,\psi(x)}$ is not a symmetry of the free field theory. The mass term is fine $m^2 \phi \phi^* \rightarrow m^2 \phi' \phi'^* = m^2 \phi \phi^*$, but

$$\partial_\mu \left(e^{i \psi} \phi \right) = e^{i \psi} \left(\phi_{,\mu} + i \psi_{,\mu} \phi \right) \tag{23.15}$$

shows us that the derivative term will not lose all ψ dependence. Here is the unusual question we now ask (and a similar issue comes up in GR, in a very similar setting) – is it possible to redefine ∂_μ such that $D_\mu \left(e^{i \psi} \phi \right) = e^{i \psi} D_\mu \phi$? Then forming the action out of this modified derivative will automatically enforce local gauge invariance.

If we define the primed fields $\phi' \equiv e^{i \psi} \phi$ and $A'_\mu \equiv A_\mu + \beta \psi_{,\mu}$ then solving for $\psi_{,\mu}$ in terms of A_μ and A'_μ, we can rewrite (23.15)

$$\begin{gathered} \partial_\mu \phi' = e^{i \psi} \left(\partial_\mu \phi + i \phi \frac{1}{\beta} \left(A'_\mu - A_\mu \right) \right) \\ \downarrow \\ \partial_\mu \phi' - \frac{i}{\beta} \phi' A'_\mu = e^{i\psi} \left(\partial_\mu \phi - \frac{i}{\beta} \phi A_\mu \right), \end{gathered} \tag{23.16}$$

which immediately suggests that $D_\mu \equiv \partial_\mu - \frac{i}{\beta} A_\mu$ is the most likely candidate – this operator is called the "covariant derivative" and will be generalized in the general relativistic setting. If we write the Lagrangian:

$$\bar{\mathcal{L}} = F^{\mu\nu} \left(A_{\nu,\mu} - A_{\mu,\nu} \right) - \frac{1}{2} F^2 + D_\mu \phi \, g^{\mu\nu} \left(D_\nu \phi \right)^* - m^2 \phi \phi^* \tag{23.17}$$

then we will get a consistent field theory, with some obvious replacements, but most importantly $\partial_\mu j^\mu = 0$. The constant β used in our D_μ is set to α^{-1} for the above. Our new derivative here is more notation than anything else, and provides a compact form for the Lagrangian. But let's be clear, when you get down to the physics of the theory, like the continuity equation, it is our familiar ∂_μ that is useful – that's the one which allows us to integrate.

23.3.1 Field Equations of the Sourced System

If we write out all the terms in (23.17) in preparation for variation, we have

$$\begin{aligned} \bar{\mathcal{L}} = & F^{\mu\nu} \left(A_{\nu,\mu} - A_{\mu,\nu} \right) - \frac{1}{2} F^2 - m^2 \phi \phi^* \\ & + \left[\phi_{,\mu} g^{\mu\nu} \phi^*_{,\nu} + \frac{i}{\beta} \phi_{,\mu} g^{\mu\nu} A_\nu \phi^* - \frac{i}{\beta} A_\mu \phi g^{\mu\nu} \phi^*_{,\nu} + \frac{1}{\beta^2} A_\mu \phi A^\mu \phi^* \right]. \end{aligned} \tag{23.18}$$

Nothing has changed for the $F^{\mu\nu}$ variation, and we recover the usual $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. The potential variation is more complicated (setting $\beta = 1/\alpha$ to get the correct limiting case):

$$\begin{aligned} \delta\bar{\mathcal{L}}(\delta A_\mu) &= \delta A_{\mu,\nu}\,(F^{\nu\mu} - F^{\mu\nu}) \\ &\quad - i\,\alpha\,(\phi\,\partial^\mu\,\phi^* - \phi^*\,\partial^\mu\,\phi)\;\delta A_\mu + 2\,\alpha^2\,A^\mu\phi\,\phi^*\,\delta A_\mu, \end{aligned} \tag{23.19}$$

and flipping the derivative on the first sign, to turn $\delta A_{\mu,\nu} \rightarrow \delta A_\mu$, we get the field equation

$$-2\,F^{\mu\nu}{}_{,\nu} = i\,\alpha\,(\phi\,\partial^\mu\,\phi^* - \phi^*\,\partial^\mu\,\phi) - 2\,\alpha^2\,\phi\,\phi^*\,A^\mu \tag{23.20}$$

and clearly, we now have the natural current definition:

$$\begin{aligned} j^\mu &\equiv i\,(\phi\,\partial^\mu\,\phi^* - \phi^*\,\partial^\mu\,\phi) - 2\,\alpha\,\phi\,\phi^*\,A^\mu \\ &= i\,(\phi\,(D^\mu\,\phi)^* - \phi^*\,D^\mu\,(\phi))\,. \end{aligned} \tag{23.21}$$

For the ϕ and ϕ^* variation, we have

$$\begin{aligned} \delta\bar{\mathcal{L}}(\delta\phi^*) &= \left(-m^2\,\phi - \partial^\mu\partial_\mu\,\phi + i\,\alpha\,\phi_{,\mu}\,A^\mu + (i\,\alpha\,A^\mu\,\phi_{,\mu} + i\,\alpha\,A^\mu{}_{,\mu}\,\phi) + \alpha^2\,A^\mu\,A^\nu\right)\,\delta\phi^* = 0 \\ &= -m^2\,\phi - D^\mu\,D_\mu\,\phi = 0. \end{aligned} \tag{23.22}$$

And we get the same thing for ϕ^* via ϕ variation (namely $(D_\mu\,D^\mu\,\phi)^* + m^2\,\phi^* = 0$). It is interesting that the effect of the move $\partial \rightarrow D$ is just replacement, we can treat it just like ∂_μ in the variation.

Finally, we want $\partial_\mu j^\mu = 0$:

$$\begin{aligned} \partial_\mu j^\mu &= i\,(\partial_\mu\,\phi\,\partial^\mu\,\phi^* + \phi\,\partial_\mu\partial^\mu\,\phi^* - \partial_\mu\,\phi^*\,\partial^\mu\,\phi - \phi^*\,\partial_\mu\partial^\mu\,\phi) \\ &\quad - 2\,\alpha\,\phi_{,\mu}\,\phi^*\,A^\mu - 2\,\alpha\,\phi\,\phi^*_{,\mu}\,A^\mu - 2\,\alpha\,\phi\,\phi^*\,A^\mu{}_{,\mu} \\ &= i\,(\phi\,(D^\mu\,D_\mu\,\phi)^* - \phi^*\,(D^\mu\,D_\mu\,\phi)) \\ &= 0 \end{aligned} \tag{23.23}$$

where the final equality holds by virtue of the field equations.