Radial Infall

Lecture 31

Physics 411
Classical Mechanics II

November 14th, 2007

Focusing on orbital and bending solutions for material particles and light (respectively) follows a natural Newtonian progression. We can also look at pure radial infall – it is interesting to note that, while physically distinct from Newtonian infall, the equations of motion for this case are . . . similar to the Newtonian ones. We will discuss radial geodesics for both massive particles and light. Of course, for light, there is no classical analogue, so we are in a difficult spot when it comes to interpretation of the radial infall of light. We will pick up this interesting question next time, but today, we make a short digression and discuss redshift.

31.1 Radial Infall

So far, we have looked at "orbital" solutions for test particles and grazing solutions for light. Radial infall is actually an easier case, corresponding to simple quadrature in Newtonian gravity, so we are going backwards up to a point. These solutions can be thought of as zero angular momentum degenerate solutions to the orbital equations of motion.

31.1.1 Massive Particles

For a time-like trajectory, with the Schwarzschild metric, the Lagrangian is

$$L = \frac{1}{2}\,\dot{x}^\mu\, g_{\mu\nu}\,\dot{x}^\nu = \frac{1}{2}\left(-\dot{t}^2\left(1 - \frac{2\,M}{r}\right) + \frac{\dot{r}^2}{1 - \frac{2\,M}{r}} + r^2\,\dot{\theta}^2 + r^2\,\sin^2\theta\,\dot{\phi}^2\right). \tag{31.1}$$

For zero angular momentum, we have $J_x = J_y = J_z = 0$ which can be used to set $\dot{\theta} = \dot{\phi} = 0$ and $\theta = \frac{1}{2}\,\pi$. The temporal angular momentum is, as always, constant, so we set $E = -\frac{\partial L}{\partial \dot{t}}$. For material particles in affine parametrization, $L = H = -\frac{1}{2}$, this is the only difference between the light and matter radial solutions. Putting it all together, we have an equation of motion for r,

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{M + r^2\,\ddot{r}}{r^2 - 2\,M} = 0 \rightarrow \ddot{r} = -\frac{M}{r^2}. \tag{31.2}$$

This is surprisingly similar (which is to say "identical") to the equation for Newtonian infall – the interpretation is slightly different – we have r the Schwarzschild radial coordinate rather than a flat spherical one, and the derivatives are w.r.t. τ, the proper time of the particle rather than t, but the forms are the same. The solution, as usual, is obtained by multiplying through by $\dot{r}$:

$$\dot{r}\,\ddot{r} = -\frac{M\,\dot{r}}{r^2} \rightarrow \frac{1}{2}\frac{d}{d\tau}\left(\dot{r}^2\right) = \frac{d}{d\tau}\left(\frac{M}{r}\right) \tag{31.3}$$

and we can directly integrate (losing a constant of integration) the above to write $\tau(r)$ or vice-versa as:

$$\boxed{\tau(r) = \frac{\pm 2}{3\sqrt{2\,M}}\left(r^{3/2} - r_0^{3/2}\right)} \tag{31.4}$$

where we "begin" at r_0. Because of the square root we took, there is a choice of sign – if we start at r_0 and fall towards $r \rightarrow 0$, then the correct sign is negative (so that time is positive, since $r_0 > r$). That's just the normal classical result, we start off at r_0 and go towards the center. What an observer at infinity sees, however, is quite a bit different. We must switch from $\tau(r)$ to $t(r)$ the coordinate (observer at infinity) time to find out how the situation has changed:

$$\begin{aligned} \frac{d\tau}{dr} &= \frac{dt}{dr}\frac{d\tau}{dt} = -\sqrt{\frac{r}{2\,M}} \\ \frac{dt}{dr} &= -\sqrt{\frac{r}{2\,M}}\left(\frac{E\,r}{2\,M - r}\right), \end{aligned} \tag{31.5}$$

(we calculate $\frac{d\tau}{dt}$ from the definition of t-momentum which is constant). This is easily integrated (by switching to $y \equiv \frac{2\,M}{r}$), and the end result for coordinate time is:

$$\boxed{\begin{aligned} t(r) = \frac{E}{3}\sqrt{\frac{2}{M}}\left(\sqrt{r}\,(6\,M + r) - \sqrt{r_0}\,(6\,M + r_0)\right) \\ + 2\,M\,E\left[\log\left(\frac{1-\sqrt{\frac{2\,M}{r}}}{1+\sqrt{\frac{2\,M}{r}}}\right) - \log\left(\frac{1-\sqrt{\frac{2\,M}{r_0}}}{1+\sqrt{\frac{2\,M}{r_0}}}\right)\right]\end{aligned}} \tag{31.6}$$

Now we can compare the two – our traveling version $\tau(r)$ which looks Newtonian, and the observed version $t(r)$. The result for $r_0 = 10$ is shown in Figure 31.1. Apparently, while we happily travel towards the center of the source, it looks externally as if we are constantly slowing down, the observer believes we are still around.

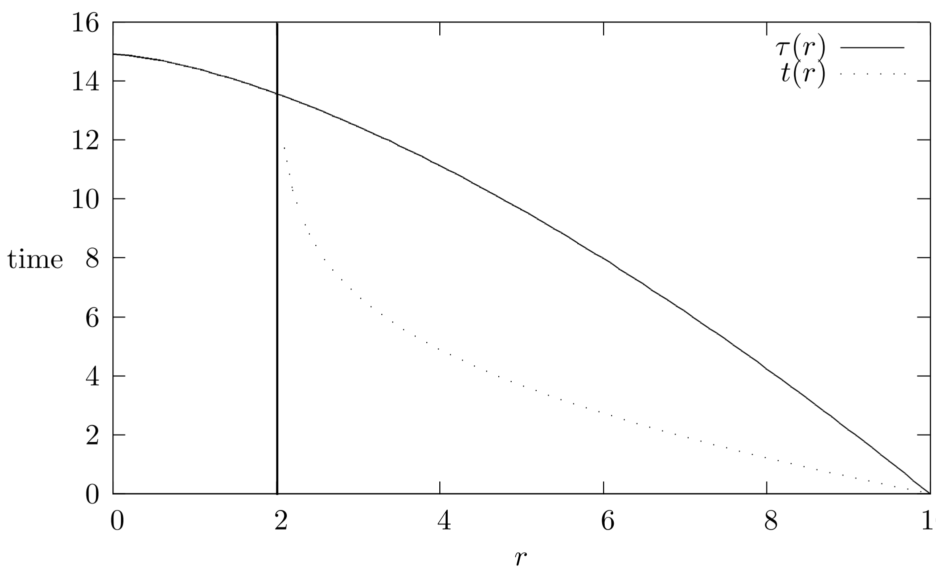

Figure 31.1: Proper time $\tau(r)$ and coordinate time $t(r)$ as a function of distance. The mass here is $M = 1$, and the asymptote $r = 2\,M$ is never reached for $t(r)$.

31.1.2 Light

Exactly the same type of analysis works for the case of light, $L = H = 0$, and we have to solve the radial equations again. If we take the usual, $J_x = J_y = J_z = 0$ solution with $L = 0$ and $\dot{t} = \frac{E\,r}{r-2\,M}$, then the equation of motion for r is just $\dot{r}^2 = E^2$. We switch to a $t(r)$ parametrization as with massive particles:

$$\begin{aligned}\frac{dt}{d\tau} &= \frac{dt}{dr}\,\frac{dr}{d\tau} = \frac{E\,r}{r-2\,M} \\ \frac{dt}{dr} &= \pm\frac{r}{r-2\,M}\end{aligned} \tag{31.7}$$

and integrating this, we have:

$$\boxed{t(r) = \mp(r + 2\,M\log(r-2\,M)) \pm (r_0 + 2\,M\log(r_0-2\,M)).} \tag{31.8}$$

We can see in Figure 31.2 the result of all this for a variety of r_0 – we have taken both the "−" solutions (these go towards the $r = 2\,M$ point) and "+" solutions (going away) on either side of the discontinuous $r = 2\,M$ point.

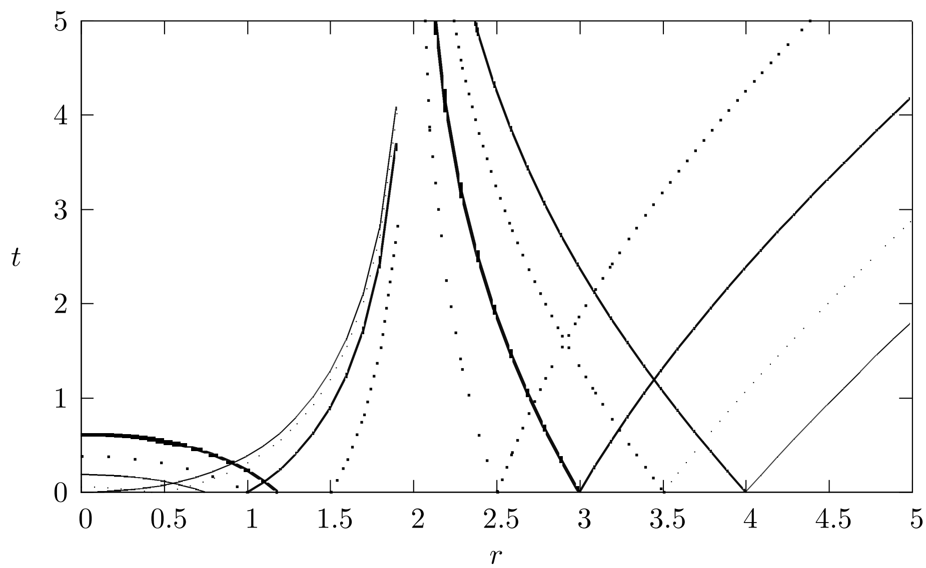

Figure 31.2: Ingoing and outgoing radial infall for light – notice the boundary at $r = 2\,M$ ($M = 1$ here).

From the form of the Schwarzschild metric, it is clear that something happens at $r = 2\,M$, the "Schwarzschild radius". As observers at infinity, we never see anything fall below this surface, while as travelers, we go right through it. Light itself is trapped inside, unable to escape, so a person caught at $r < 2M$ would not have contact with the outside universe. Such a surface is called an "event horizon". In fact, our intuition about light, to the extent that we have it, is incomplete in this picture – it is possible to show that light can cross the event horizon, but cannot leave. The issue with the picture in Figure 31.2 is effectively the lack of an analagous proper time which would allow us to see the trajectory of the light in its own frame.

31.2 Gravitational Redshift in Schwarzschild Geometry

As a warmup for the next section – let's consider the redshift of light – we will emit light from our lab at rest at r, to be observed by a lab (at rest) very far away at infinity. What does this mean physically? Well, first consider our geodesic – as material particles, we obey:

$$-\frac{1}{2} = L = \frac{1}{2}\left(-\left(1-\frac{2\,M}{r}\right)\dot{t}^2 + \left(1-\frac{2\,M}{r}\right)^{-1}\dot{r}^2 + r^2\,(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right), \tag{31.9}$$

and we put our own motion in some equatorial plane $J_x = J_y = 0$. If, in addition, we require that $\dot{\theta} = \dot{\phi} = 0$, indicating that we *are* indeed at rest (through whatever agency), then our four-velocity $\dot{x}^\alpha$ is precisely given by solving the above for $\dot{t}$:

$$-\frac{1}{2} = -\frac{1}{2}\left(1-\frac{2\,M}{r}\right)\dot{t}^2 \rightarrow \dot{t} = \left(1-\frac{2\,M}{r}\right)^{-\frac{1}{2}} \tag{31.10}$$

so that

$$\dot{x}_r^\alpha \doteq \begin{pmatrix} \left(1-\frac{2\,M}{r}\right)^{-\frac{1}{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{31.11}$$

We are sitting at some Schwarzschild distance r, and this velocity represents the (normalized) tangent to our geodesic.

Now we emit a photon of energy $\hbar\,\omega_r$ in our frame – what does that mean? Well, the photon's four-momentum has its energy as the zero component. In our laboratory, we have a natural set of spatial basis vectors, and our own four-velocity is tangent to our geodesic in the temporal direction, so provides the unit "time" basis vector. Our measurement of the photon energy proceeds by taking the component of its momentum in our local basis, that is:

$$\hbar\,\omega_r = \dot{x}_r^\alpha\,p_\alpha = \left(1 - \frac{2\,M}{r}\right)^{-\frac{1}{2}}\,p_t. \tag{31.12}$$

The photon travels off to infinity – remember that as the Schwarzschild radial coordinate $r \to \infty$, the metric becomes Minkowski, so at infinity, a stationary observer is described by the four-velocity:

$$\dot{x}_\infty^\alpha \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{31.13}$$

and the observer at infinity measures the photon w.r.t. the temporal basis vector out there (i.e. $\dot{x}_\infty^\alpha$):

$$\hbar\,\omega_\infty = \dot{x}_\infty^\alpha\,p_\alpha = p_t. \tag{31.14}$$

From our discussion of the geodesics of light (or anything else, for that matter) – we know, by virtue of the cyclic nature of t in the Hamiltonian governing geodesics, that p_t is conserved (it's what we traditionally call $-E$) – so

$$\boxed{\hbar\,\omega_\infty = \left(1 - \frac{2\,M}{r}\right)^{\frac{1}{2}}\,\hbar\,\omega_r} \tag{31.15}$$

and the observed frequency ω_∞ is thus less than the observed emitted frequency ω_r by a familiar factor.

31.3 Infall & Coordinates

We see that the frequency of light observed at infinity (where we normally are, at least w.r.t. the fixed stars) was different from the frequency measured locally (where we normally are as we orbit about the sun). This so-called "redshift" is caused by clocks in stationary frames – the time coordinate of Schwarzschild coordinates tells us how to relate our own proper time with observation.

We will be looking more at the issue of time for photon trajectories. Remember that the line element for light is given by $ds^2 = 0$. This means that the trajectory has tangent vector $\dot{x}^\alpha$ that is null ($\dot{x}^\alpha \dot{x}_\alpha = 0$), and leads to some difficulties of interpretation (and interpretation only – we still know how to generate a well-defined trajectory, of course) for the parameter τ that shows up in the description of the equations of motion. It also means, via our discussion last time, that photons have difficulty setting up an orthonormal basis with which to measure ("red" cannot make a laboratory).

What I will suggest is that the coordinate system we have chosen does not represent the situation entirely accurately. We found, with real materials, a discrepancy between an observer "falling" towards the horizon and an observer watching the fall. From the faller's point of view, everything is fine (so to speak) until the final singularity at the center of Schwarzschild coordinates is encountered. From the point of view of the observer at infinity, the faller slows and never crosses the horizon at all. This seeming paradox was neatly resolved by shifting attention from the proper time to the coordinate time.

With light, there is only one physically meaningful "time" – coordinate. Yet, we imagine the same basic situation should hold: light should cross unperturbed into the inner region (i.e. cross the horizon), while an observer should see it getting slower and slower (redder and redder) without ever going away. One of these directions was provided by coordinate time, but we need to get a handle on the other. Without a natural proper time, the analysis gets a bit trickier.

31.4 Transformation

So far, we have been examining the Schwarzschild solution in a single coordinate system. That's equivalent to studying classical mechanics in Cartesian coordinates only. The advantage, of course, is that these coordinates are well-adapted to the spherical symmetry of the space-time(meaning that the angular Killing tensors retain their Euclidean form). They do not reveal all aspects of the geodesics, though. For example, the metric written in

Schwarzschild coordinates has line element

$$ds^2 = -\left(1 - \frac{2\,M}{r}\right)\,dt^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\,dr^2 + r^2\left(d\theta^2 + \sin^2\,\theta\,d\phi^2\right), \tag{31.16}$$

and for $r > 2\,M$, the dt component (g_{00}) is always negative, allowing us to make the association with a time coordinate, if only by comparison to Minkowski. For $r < 2\,M$, it is the radial g_{rr} that is negative, so evidently inside the event horizon, the metric looks more like Minkowski with r playing the role of time. What's worse, the metric *at* $r = 2\,M$ isn't even well-defined.

What's going on here? We know that $r = 2\,M$ has little effect on the motion of particles. There is strange behavior there from our viewing station at infinity, but a particle itself passes right through $r = 2\,M$ without even realizing GR exists. In fact, this coordinate singularity (as it turns out) is very much akin to the singularity at the poles of the two-dimensional spatial metric:

$$ds^2 = r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \tag{31.17}$$

where the length of vectors at $\theta = 0$, and π is effectively undetermined. That's easy enough to handle there, we just re-coordinatize so the pole is somewhere else and happily continue to measure. What is the equivalent procedure for Schwarzschild coordinates?

The problem is really with the two-dimensional (t, r) subspace, so our goal is to find new coordinates (u, v) that are explicitly continuous. Clearly, this will involve a mixing of the original (t, r), and we take the simplest possible ansatz (although others are available, and preferable depending on the setting). This will lead us to a precisely linear (in M) representation of the metric – i.e. in the coordinates we are about to develop, the full Schwarzschild metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + \frac{M}{r}\,h_{\mu\nu}$ with $h_{\mu\nu}$ of order 1. This is a simple example of "analytic extension", the process by which we attempt to cover as much of space-time as possible with a single coordinate system.

Using a principle of minimal violence, consider the two dimensional portion of the Schwarzschild metric

$${}^2ds^2 \equiv -\left(1 - \frac{2\,M}{r}\right)\,dt^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\,dr^2 \tag{31.18}$$

and define new coordinates via $u = t + f(r)$, $v = r$. The motion for this choice is simple – because ${}^2ds^2$ is independent of time, only the element dt

transforms, and we don't have to worry about complicated implicit definitions of v in terms of r. We are allowing t to change by any function of r in a separable manner (additive separation). We demand that in the (u, v) coordinates, the line element take the form

$$ {}^2ds^2 = -\left(1 - \frac{2\,M}{r}\right)\,du^2 + \left(1 + \alpha\,\frac{M}{r}\right)\,dv^2 + \beta\,\frac{M}{r}\,du\,dv, \tag{31.19}$$

again a minimalist approach – requiring that the new "time" coordinate u play the same role as t in ${}^2ds^2$ which is already linear in $\frac{M}{r}$ and asking that the remaining portion of the metric also separate into Minkowski plus a linear correction. Then inputting the ansatz, we have

$$\begin{aligned} {}^2ds^2 &= -\left(1 - \frac{2\,M}{r}\right)\,du^2 + \left(\left(-1 + \frac{2\,M}{r}\right)\,f'^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\right)\,dv^2 \\ &\quad + 2\left(1 - \frac{2\,M}{r}\right)\,f'\,du\,dv \\ &= -\left(1 - \frac{2\,M}{r}\right)\,du^2 + \alpha\,\frac{M}{r}\,dv^2 + \beta\,\frac{M}{r}\,du\,dv. \end{aligned} \tag{31.20}$$

We can solve the cross-term equation $2\left(1 - \frac{2\,M}{r}\right)\,f' = \beta\,\frac{M}{r}$ easily, this gives

$$ f' = \frac{\beta M}{2\,(r - 2\,M)} \longrightarrow f(r) = f_0 + \frac{1}{2}\,\beta\,M\,\log(r - 2\,M)\,, \tag{31.21}$$

and introduces an integration constant f_0. Turning to the dr^2 equation,

$$ \left(\left(-1 + \frac{2\,M}{r}\right)\,f'^2 + \left(1 - \frac{2\,M}{r}\right)^{-1}\right) = \frac{\left(r - \frac{1}{2}\,\beta\,M\right)\left(r + \frac{1}{2}\,\beta\,M\right)}{r\,(r - 2\,M)} \tag{31.22}$$

and we can get the desired form by setting $\beta = 4$, at which point we see that $\alpha = 2$. The final solution is

$$ {}^2ds^2 = -\left(1 - \frac{2\,M}{r}\right)\,du^2 + \left(1 + \frac{2\,M}{r}\right)\,dr^2 + \frac{4\,M}{r}\,du\,dr \tag{31.23}$$

and the new "time" coordinate u is related to t and r via:

$$ u = t + (f_0 + 2\,M\,\log(r - 2\,M))\,. \tag{31.24}$$

The full metric is given by the line element

$$ \boxed{ds^2 = -\left(1 - \frac{2\,M}{r}\right)\,du^2 + \left(1 + \frac{2\,M}{r}\right)\,dr^2 + \frac{4\,M}{r}\,du\,dr + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right)} \tag{31.25}$$

which is free of any obvious singularities except at $r = 0$ – this metric is in Eddington coordinates. We can once again calculate the radial null geodesics using this new coordinate system. As before, we set the angular momenta to zero (to get the radial portion), and the Lagrangian to zero (to get null geodesics, appropriate to a description of light). The equations for $\dot{u}$ and $\dot{r}$ become

$$\begin{aligned} \dot{u} &= E \quad \dot{r} = -E \\ \dot{u} &= \frac{E\,(r + 2\,M)}{r - 2\,M} \quad \dot{r} = E, \end{aligned} \tag{31.26}$$

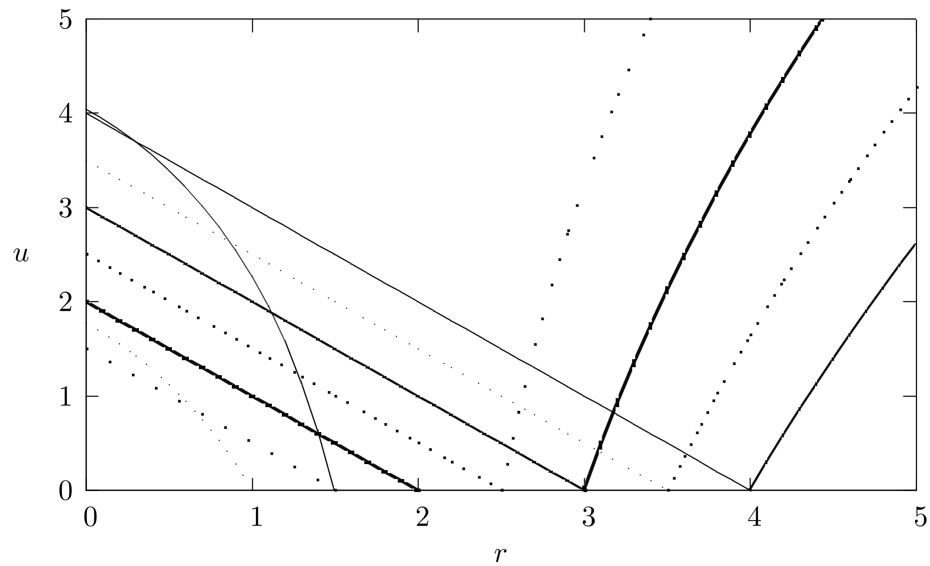

Figure 31.3: Radial infall of light for the Eddington metric.

and switching to the $u(r)$ parametrization we get the two solutions

$$\begin{aligned} u(r) &= r_0 - r \\ u(r) &= r_0 + r + 4\,M\,\log(r - 2\,M) \end{aligned} \tag{31.27}$$

shown in Figure 31.3. From the geodesics here, it is clear that light-like particles can cross the horizon at $r = 2\,M$, unlike the equivalent picture in Schwarzschild coordinates which indicates that the light never reaches the horizon. In neither case can the light emerge from the interior, cutting off information from $r < 2\,M$ from the outside world.

The procedure we have carried out in this section makes the physical picture for radial infall of both light and point particles clear – from the particle's point of view, the fall is "just like" Newtonian gravity (proper time) yet an observer would see the particle slowing down and never reaching the $r = 2M$ horizon (coordinate time). The case for light is slightly more involved because there is no notion of proper time for a photon – to discuss the photon's point of view, we must transform the metric, effectively changing what we mean by the temporal coordinate – then we see that once again, the photon goes right through the horizon (coordinate time in Eddington metric) and from an observer at infinite, the photon never reaches the horizon (coordinate time in Schwarzschild metric).

In addition to making the connection between the two points of view (observers traveling with different types of particles, and observers at infinity), the Eddington form of the metric is manifestly free of discontinuity, and (most surprisingly, perhaps) is linear in the source mass M.