Final Physics of Schwarzschild

Lecture 32

Physics 411
Classical Mechanics II

November 16th, 2007

We have studied a lot of properties of the Schwarzschild metric – I want to finish with a "fun" calculation, and a short discussion of the difference between classical Newtonian theory and the GR machinery we have introduced. Keep in mind the rough idea – radial directions are "unchanged" (in form) in GR, although the interpretation of them is quite different. It is orbital motion which introduces a natural "force" interpretation different from our usual one (indeed, we saw that "gravito-magnetic" forces can show up), radial motion, both for light and materials is relatively similar between the theories.

32.1 Physical Notions

We've been discussing this very special location, $r = 2\,M$, and it is generic in the sense that any spherically symmetric space-time has one. In particular, the earth has one at ≈ 1 cm, the sun has one at ≈ 3 km, everything has a distance $r = 2\,M$ from its center, so why don't we observe any of these crazy effects? Because the Schwarzschild radius is *inside* the object, where Einstein's equation is not source-free – we're using the wrong solution to Einstein's equation[1]. There is a very special class of objects in which the Schwarzschild radius is *outside* the matter distribution, these are black holes, the GR equivalent of point particles. We saw that light (a.k.a. information) cannot escape the "boundary" (event horizon) at $r = 2\,M$, hence the name.

Currently, black holes are known to exist, although this is a relatively recent development – they have been a theoretical prediction since the

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[1] Just as you cannot use an exterior solution for a uniformly charged sphere inside the sphere in E&M.

Schwarzschild solution was introduced. Some of the properties of black holes are not as exotic as they sound – for example, we can ask in classical gravity what the escape velocity of light is. The escape velocity is defined to be the minimum velocity of a particle such that the kinetic energy of the particle overcomes the gravitational energy. In our units, the defining equation is:

$$\frac{1}{2}\,v^2 = \frac{M}{r}, \tag{32.1}$$

using this we can ask what is the radius corresponding to a particular escape velocity. For light, $v = c = 1$ (again, in our units), so that the radius associated with the entrapment of light is $r = 2\,M$. That is, we would, even classically, expect that for a mass M, light is trapped inside a spherical surface of Schwarzschild radius.

32.1.1 Real Material Infall

Previously, we discussed a test particle falling freely under the influence of Schwarzschild geometry. This is an artificial situation, for non-point particles the question of what happens as one travels towards the Schwarzschild radius, or singularity at $r = 0$ is nonsensical, since you never make it there. I'm not talking about any deep physical property – you die relatively far away from the source. Let's see how this happens[2].

What we really want to know is the deviation of two different parts of the body as we fall freely – at some point, our feet are accelerated faster than our heads, and there is an overall stretching. Sounds like a job for the equation of geodesic deviation, which is, again

$$\frac{D^2}{D\tau^2}\,\eta^\alpha = -R^\alpha{}_{\rho\gamma\beta}\,\dot{x}^\beta\,\dot{x}^\rho\,\eta^\gamma, \tag{32.2}$$

where $\dot{x}^\beta$ is the geodesic four-velocity, so that $\dot{x}^\gamma\,\dot{x}^\beta{}_{;\gamma} = 0$, since $x^\beta(\tau)$ is a geodesic. The separation η^γ is a four-separation, we'll take care of that in a moment. The idealization we have in mind: there are single particles representing your head and feet, and we'll calculate the relative acceleration between them.

Our two points are traveling along (particle) radial geodesics, so from our discussion of point particle geodesics, we know the geodesic form immedi-

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[2] Don't be offended, I am not trying to be morbid *or* cute, this calculation brings up some useful ideas.

ately – all we need are the velocities – this expression comes from assuming we start at infinity with zero velocity and head in (how do you know?):

$$\dot{x}^{\mu} \doteq \begin{pmatrix} \frac{1}{1-\frac{2\,M}{r}} \\ -\sqrt{\frac{2\,M}{r}} \\ 0 \\ 0 \end{pmatrix}. \tag{32.3}$$

Using this and the Riemann tensor appropriate to the Schwarzschild metric. We can calculate the right-hand side of (32.2),

$$\frac{D^2}{D\tau^2}\,\eta^{\gamma} \doteq \begin{pmatrix} -\frac{4\,M^2}{(r-2\,M)\,r^3} & -\frac{2\,M\sqrt{2\,M}}{(r-2\,M)^2\,r^{3/2}} & 0 & 0 \\ \frac{2\,M\sqrt{2\,M}}{r^{3/2}} & \frac{2\,M}{(r-2\,M)\,r^2} & 0 & 0 \\ 0 & 0 & -\frac{M}{r^3} & 0 \\ 0 & 0 & 0 & -\frac{M}{r^3} \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta r \\ \Delta\theta \\ \Delta\phi \end{pmatrix}, \tag{32.4}$$

where the deviation vector η^{γ} is expressed as the vector on the right above.

Okay, so here's the problem – first, the time and radial portions are mixed, and second, we aren't in an orthonormal basis. What we're going to do is change bases, so we need to pick an orthonormal "frame". We haven't discussed bases much, but they really aren't so different here than in flat space – we pick, somewhere along the geodesic, four orthogonal vectors and parallel transport them around with us (they will thus *remain* orthogonal). In this case, since we want to do a local calculation of forces in our local frame, we need to construct four vectors $\{\hat{\mathbf{e}}^{\alpha}_{\mathbf{0}}, \hat{\mathbf{e}}^{\alpha}_{\mathbf{r}}, \hat{\mathbf{e}}^{\alpha}_{\theta}, \hat{\mathbf{e}}^{\alpha}_{\phi}\}$ with the following properties:

$$\boxed{\begin{aligned} -1 &= \hat{\mathbf{e}}^{\alpha}_{\mathbf{0}}\,\hat{\mathbf{e}}^{\beta}_{\mathbf{0}}\,g_{\alpha\beta} \\ \delta_{ij} &= \hat{\mathbf{e}}^{\alpha}_{\mathbf{i}}\,\hat{\mathbf{e}}^{\beta}_{\mathbf{j}}\,g_{\alpha\beta} \\ 0 &= \frac{D}{D\tau}\,\hat{\mathbf{e}}^{\alpha}_{\mathbf{0}} = \frac{D}{D\tau}\,\hat{\mathbf{e}}^{\alpha}_{\mathbf{i}}. \end{aligned}} \tag{32.5}$$

The first two lines just tell us that the basis vectors, at each point, are orthogonal and normalized (in the Minkowski sense). The third line is the parallel transport statement for all four. Don't let all the bold-face get in the way, these are just contravariant vectors that we interpret as a basis, which just means we will project equations onto them. The basis vectors describe a lab frame moving with one of the particles.

There is still the question of choice – in theory the above do not determine four unique vectors. But Schwarzschild, with its structure of $M \times S^2$ (M

refers to the two-dimensional $t-r$ space, and S^2 is the 2-sphere, just a fancy way of pointing out that the deviation from spherical Minkowski is in the r and t directions) has angular unit vectors built-in, i.e. take;

$$\hat{\mathbf{e}}_\theta \doteq \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} \qquad \hat{\mathbf{e}}_\phi \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r\,\sin\theta} \end{pmatrix} \tag{32.6}$$

just as in, for example, Griffiths. We still need a vector for time and the radial coordinate – but we have $\dot{x}^\mu\,\dot{x}_\mu = -1$ with $\dot{x}^\mu$ parallel-transported along x^μ all by construction of the radial geodesic. For that reason alone, we might set $\hat{\mathbf{e}}^\alpha_{\mathbf{0}} = \dot{x}^\alpha$. More importantly, this choice puts us in our rest frame – we measure our velocity to be zero (instantaneously, the basis vectors change as we move of course). Mathematically, this has the effect of setting the time-separation η^0 to zero so we can measure spatial acceleration simultaneously. With three of the four basis vectors fixed, the fourth is actually unique – using the definition of basis vectors from above, and specializing to the $t-r$ plane, the radial basis vector has the form:

$$\hat{\mathbf{e}}_{\mathbf{r}} \doteq \begin{pmatrix} \frac{\sqrt{2\,M\,r}}{2\,M-r} \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{32.7}$$

All right, that's all very well and good, we have constructed a suitable basis, but now we have to transform the deviation equation. This is not so bad, using the transport property of the basis vectors[3],

$$\boxed{g_{\alpha\gamma}\,\hat{\mathbf{e}}^\alpha\,\frac{D}{D\tau}\,\eta^\gamma = \frac{D}{D\tau}\,(g_{\alpha\gamma}\,\hat{\mathbf{e}}^\alpha\eta^\gamma)} \tag{32.8}$$

for each of the $\hat{\mathbf{e}}$. So we can *define* the projected deviation vector by

$$\begin{aligned} \Delta t' &= \hat{\mathbf{e}}^\alpha_{\mathbf{0}}\,g_{\alpha\beta}\,\eta^\beta \\ \Delta r' &= \hat{\mathbf{e}}^\alpha_{\mathbf{r}}\,g_{\alpha\beta}\,\eta^\beta \\ \Delta\theta' &= \hat{\mathbf{e}}^\alpha_\theta\,g_{\alpha\beta}\,\eta^\beta \\ \Delta\phi' &= \hat{\mathbf{e}}^\alpha_\phi\,g_{\alpha\beta}\,\eta^\beta. \end{aligned} \tag{32.9}$$

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[3] Basically, what we have done is diagonalize the right-hand side of the deviation equation with eigenvectors that are parallel-transported (read "constant" w.r.t. the $\frac{D}{D\tau}$ operation), then we can do the usual – redefine η on both sides of the equation so as to diagonalize the system. In this setting, the eigenvalues of the matrix on the right are 0, $\frac{2\,M}{r^3}$ and $-\frac{M}{r^3}$, precisely what we get through the basis transformation).

Contracting each of our four deviation equations with the four basis vectors, and using the transformation rule above gives us the following set of four equations:

$$\begin{aligned}\frac{D^2}{D\tau^2}\,(\Delta t') &= 0\\ \frac{D^2}{D\tau^2}\,(\Delta r') &= \frac{2\,M}{r^3}\,\Delta r'\\ \frac{D^2}{D\tau^2}\,(\Delta\theta') &= -\frac{M}{r^3}\,\Delta\theta'\\ \frac{D^2}{D\tau^2}\,(\Delta\phi') &= -\frac{M}{r^3}\,\Delta\phi',\end{aligned} \tag{32.10}$$

where the first equation comes from choosing $\hat{\mathbf{e}}^{\alpha}_{\mathbf{0}} = \dot{x}^{\alpha}$, and tells us that we will be measuring events simultaneously in this frame. We are primarily concerned with the radial equation, the radial acceleration of the two particles (head and feet) is what will cause the tension.

To go further, we need to refine our idealization – point particles at the head and feet is too coarse to get a material tension – let's use the standard: the human body is approximated as a rectangular box with height h, and width w, we take as our total mass m, then $\rho = \frac{m}{h\,w^2}$ is our density. The relative acceleration of two nearby parcels of mass, viewed as a force, is counteracted by our bodies, so that

$$d\mathbf{F}' = dm\,\frac{2\,M}{r^3}\,\Delta r'\,\hat{\mathbf{e}}_{\mathbf{r}}, \tag{32.11}$$

with the rest frame set at the center of mass, so $\Delta r'$ is just the distance to the center of mass of the body. What is the net force across the square cross-section at the center? This is what we need to calculate to get the pressure (force per unit area). Using $dm = \rho\,dV$, we have

$$\mathbf{F}' = \hat{\mathbf{e}}_{\mathbf{r}} \int_{-\frac{h}{2}}^{0} \frac{2\,M}{r^3}\,r'\,(\rho\,w^2\,dr') = \hat{\mathbf{e}}_{\mathbf{r}} \int_{-\frac{h}{2}}^{0} \frac{2\,M\,r'}{r^3}\,\frac{m}{h}\,dr' = -\frac{M\,m\,h}{4\,r^3}\,\hat{\mathbf{e}}_{\mathbf{r}}, \tag{32.12}$$

that's the net force on the center of mass, to get pressure (tension, really, because negative), we just divide by w^2:

$$P = -\frac{|\mathbf{F}'|}{w^2} = -\frac{M\,m\,h}{4\,w^2\,r^3}. \tag{32.13}$$

All right, time to put in the numbers – suppose we assume the black hole has solar mass, $M \approx 1.5$ km, that we have a width of 50 cm, a height of 2

m. For our mass, take 75 kg $\approx 5.6 \times 10^{-29}$ km, then we just have to convert pressure into appropriate units.

$$\begin{aligned} 1\text{Atm} &= 101,325 \text{ Pa} = (101,325)\frac{\text{kg}}{\text{m s}^2} \\ &= 101,325 \times \frac{G}{c^4}\frac{\text{kg}}{\text{m s}^2} = 8.346 \times 10^{-40}\frac{1}{\text{m}^2} \times \left(\frac{1000 \text{ m}}{1 \text{ km}}\right)^2 \\ &= 8.346 \times 10^{-34}\frac{1}{\text{km}^2} \end{aligned} \tag{32.14}$$

which has the same units as our pressure formula. The human body can withstand ≈ 100 Atm of pressure, so we solve for the radius:

$$\begin{aligned} -\frac{(1.5 \text{ km})(5.6 \times 10^{-29} \text{ km})(.002 \text{ km})}{4(.0005 \text{ km})^2 r^3} &= 100(8.346 \times 10^{-34})\frac{1}{\text{km}^2} \\ r &\approx 130 \text{ km}. \end{aligned} \tag{32.15}$$

Maybe not the most enlightening number ever (and you will be incensed when you find out the Newtonian result), the point is, you can't get very close. What is more important is the procedure – we used the equation of geodesic deviation to calculate the deviation between two points on nearby geodesics. In order to connect this with our ideas about three-dimensional space, we required that the separation vector be purely spatial (choice of basis), so that we measured distances between geodesics at equal (coordinate) time. In this subspace, we also used an orthonormal basis rather than the un-normalized "coordinate" basis, basically ensuring that our notion of integration was preserved from spherical coordinates. Then we interpreted the deviation caused by the geometry of space-time as an acceleration that would cause *us* to feel a force. This is typical of local measurements: Go to the rest frame of the observer to find out what is happening "in the lab".

Why haven't we had to set basis vectors until now? Most of the tests we have discussed involved observing the behavior of light and particles from afar – at a viewing platform located far away from the sources, where space-time is effectively flat (asymptotic limit of the Schwarzschild solution is one way to see it, or else our work on linearized gravity where the linearization is on a Minkowski background). Even when we did make local measurements, they were scalars, and of course, no basis can change a scalar.