Gravitational Radiation

Lecture 33

Physics 411
Classical Mechanics II

November 19th, 2007

While we have not exhausted the vacuum solutions, we are about half-way done with the physically interesting ones. So it is time for a brief interlude – the non-Vacuum solutions. These fall into two general categories: waves and matter-coupling. In the linearized theory, as was mentioned in our perturbative expansion of Einstein's equation, we have a generic wave equation with and without sources – the solution to this in the tensor setting is well-defined, superimposed (because we are in the linearized limit) waves of definite frequency and two polarizations. We will be studying both the vacuum and source-driven waves predicted by GR. Gravitational waves are the basis for detection in a number of experiments, both current and planned, so it is important to understand the nature of this radiation, and the patterns generated by typical sources. Basically, though, the tools are identical to E&M with one extra polarization. The *interpretation* is somewhat different. Because this is GR, and because the "field" that radiates is really the linear part of the metric, these waves are often romantically referred to as "ripples in space-time".

Another class of non-vacuum solutions involve approximations to various fluid stress tensors – there is a nice symmetry between the wave solutions (approximation on the left of Einstein's equation), and "cosmological" solutions (approximation on the right). We will consider the basic cosmological assumptions and solutions, and introduce the cosmological constant as our vacuum solution on the matter side.

We must study vacuum radiation in order to study radiative sources, and matter in the universe begs the question of a balancing extra term in an empty universe. So it goes.

33.1 The Linearized Wave Equation

When we discussed the Newtonian correspondence of GR, we took a linearized approximation to the full Einstein equations, chose a gauge and found that the metric perturbations, $\bar{h}_{\mu\nu}$, satisfied the equation:

$$\boxed{\begin{aligned} \bar{G}_{\mu\nu} &= -\frac{1}{2}\,\partial_\rho\partial^\rho\,\bar{h}_{\mu\nu} \\ \bar{h}_{\mu\nu} &\equiv h_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,h \\ \partial^\rho\bar{h}_{\nu\rho} &= 0, \end{aligned}} \tag{33.1}$$

with the last of the above representing our gauge choice for the coordinates. The linearized equation, in the absence of sources is just:

$$\partial^\rho\partial_\rho\,\bar{h}_{\mu\nu} = 0 \tag{33.2}$$

a four-dimensional wave equation for $\bar{h}_{\mu\nu}$ written on a Minkowski background (raise and lower with $\eta_{\mu\nu}$). Since $\bar{h}_{\mu\nu}$ is symmetric, this is just ten times a scalar wave equation. We take the usual plane wave ansatz:

$$\bar{h}_{\mu\nu} = P_{\mu\nu}\,e^{i\,k_\alpha\,x^\alpha}, \tag{33.3}$$

interpret $P_{\mu\nu}$ as the polarization *tensor*, and k_μ as the wave vector – just a poor man's Fourier transform. What does this mean in terms of (33.1)? Put it in, we have

$$\begin{aligned} \partial^\rho\partial_\rho\,\bar{h}_{\mu\nu} &= \left(-\partial_0^2 + \partial_i\partial^i\right)\,P_{\mu\nu}\,e^{i\,(-k_0\,x^0 + k_j\,x^j)} \\ &= -\left(-k_0^2 + k_jk^j\right)\,\bar{h}_{\mu\nu} = -(k_\gamma\,k^\gamma)\,\bar{h}_{\mu\nu} = 0 \end{aligned} \tag{33.4}$$

and assuming the metric perturbation itself is non-zero, this requires that $k_\mu\,k^\mu = 0$. As a Fourier transform, we recognize the frequency k_0 and wave-vector k_i, and the condition $k_\mu\,k^\mu = 0$ tells us that we have a null vector, i.e. the solution is light-like, just as in E&M. In this flat space setting, we can make the identification of a field with characteristic velocity c.

The gauge condition (coordinate choice, the third equation in (33.1)) puts constraints on $P_{\mu\nu}$:

$$\begin{aligned} \partial^\rho\,\bar{h}_{\nu\rho} &= \partial^\rho\left(P_{\nu\rho}\,e^{i\,k_\mu\,x^\mu}\right) \\ &= i\,P_{\nu\rho}\,(i\,k^\rho)\,e^{i\,k_\mu\,x^\mu} = i\,k^\rho\,\bar{h}_{\nu\rho} = 0, \end{aligned} \tag{33.5}$$

and this is a Poynting-vector-like statement: the propagation of the wave, k^ρ is orthogonal to the perturbation $\bar{h}_{\nu\rho}$. In addition to this interpretation, we have constrained four of the ten components of $P_{\nu\rho}$ by requiring $P_{\nu\rho}\, k^\nu = 0$.

We're not quite done. Remember the gauge choice we made, $x^\mu \to x'^\mu + f^\mu$, inducing $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + f_{(\mu,\nu)}$, and we set:

$$\partial_\rho \partial^\rho f_\nu = -\partial^\rho \bar{h}_{\nu\rho}, \tag{33.6}$$

but to f_ν, we can add any vector w_ν satisfying $\partial^\rho \partial_\rho w_\nu = 0$ without changing the condition that $\partial^\rho \bar{h}'_{\nu\rho} = 0$ (I'm using primes once again here, otherwise this is nonsense).

So let's finish fixing the gauge, we will choose w_ν to make a further coordinate transformation inducing yet another transformation in $h_{\mu\nu}$

$$h''_{\nu\rho} = h'_{\nu\rho} - w_{(\nu,\rho)} \tag{33.7}$$

where the primes refer to the partially gauge fixed coordinates (our original choice to set $\partial^\rho \bar{h}'_{\mu\rho} = 0$), and the double primes are our new, further, choice. Since w_ν itself satisfies the wave equation, we may as well take it to be of the form:

$$w_\nu = Q_\nu\, e^{i\, k_\mu\, x^\mu} \tag{33.8}$$

with the same wave-vector k_μ we are using for the metric. Then the new (new) metric perturbation reads:

$$\begin{aligned} h''_{\nu\rho} &= h'_{\nu\rho} - i\, e^{i\, k_\mu\, x^\mu}\, (Q_\nu\, k_\rho + Q_\rho\, k_\nu) \\ &= \bar{h}'_{\nu\rho} - \frac{1}{2}\, \eta_{\nu\rho}\, \bar{h}'^{\alpha}_{\alpha} - i\, e^{i\, k_\mu\, x^\mu}\, (Q_\nu\, k_\rho + Q_\rho\, k_\nu) \\ &= \left(P_{\nu\rho} - \frac{1}{2}\, \eta_{\nu\rho}\, P^\alpha_\alpha - i\, (Q_\nu\, k_\rho + Q_\rho\, k_\nu) \right) e^{i\, k_\mu\, x^\mu} \end{aligned} \tag{33.9}$$

We have four unknowns in the above, the Q_ν. This gives the correct counting: We started with ten unknown components in the polarization tensor $P_{\nu\rho}$, we fixed four with the original gauge choice, and now we fix four more leaving two independent components, just right for two separate polarizations. So we need four equations to fix Q_ν, which four to take? The most convenient constraint is to eliminate the time-space portion of the metric, that will leave us with just the spatial section. That is, we want $h''_{0\rho} = 0$. Written out in components, the condition is

$$\begin{aligned} \rho = 0 &: 2\, Q_0\, k_0 = -i\, (P_{00} + \frac{1}{2}\, P^\alpha_\alpha) \\ \rho = j &: Q_0\, k_j + Q_j\, k_0 = -i\, P_{0j}, \end{aligned} \tag{33.10}$$

or in matrix form:

$$\underbrace{\begin{pmatrix} 2\,k_0 & 0 & 0 & 0 \\ k_1 & k_0 & 0 & 0 \\ k_2 & 0 & k_0 & 0 \\ k_3 & 0 & 0 & k_0 \end{pmatrix}}_{\equiv K} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = -i \begin{pmatrix} P_{00} + \frac{1}{2}\,P^{\alpha}_{\alpha} \\ P_{01} \\ P_{02} \\ P_{03} \end{pmatrix}. \tag{33.11}$$

The matrix on the left is nonsingular, so we can invert it, it's not even that bad:

$$K^{-1} = \begin{pmatrix} \frac{1}{2\,k_0} & 0 & 0 & 0 \\ -\frac{k_1}{2\,k_0^2} & \frac{1}{k_0} & 0 & 0 \\ -\frac{k_2}{2\,k_0^2} & 0 & \frac{1}{k_0} & 0 \\ -\frac{k_3}{2\,k_0^2} & 0 & 0 & \frac{1}{k_0} \end{pmatrix}, \tag{33.12}$$

and the solution for the coefficients Q_ν is:

$$\begin{aligned} Q_0 &= -\frac{i}{2\,k_0}\left(P_{00} + \frac{1}{2}\,P^{\alpha}_{\alpha}\right) \\ Q_j &= \frac{i}{2\,k_0^2}\left(P_{00} + \frac{1}{2}\,P^{\alpha}_{\alpha}\right)\,k_j - \frac{i}{k_0}\,P_{0j}. \end{aligned} \tag{33.13}$$

We are done, we have formed a metric perturbation that is "transverse", here meaning that there are no components except for spatial ones so that $k^\rho\,h_{\nu\rho} = 0$ refers to spatial orthogonality, as in E&M. There is an additional property of this gauge, consider the trace of the $h''_{\mu\nu}$ from its definition (33.7),

$$\begin{aligned} h''^{\mu}_{\mu} &= P^{\mu}_{\mu} - 2\,P^{\mu}_{\mu} - 2\,i\,Q_\mu\,k^\mu \\ &= -P^{\mu}_{\mu} - 2\,i\left[\frac{i}{2}\left(P_{00} + \frac{1}{2}\,P^{\mu}_{\mu}\right) + \frac{i}{2\,k_0^2}\left(P_{00} + \frac{1}{2}\,P^{\mu}_{\mu}\right)\,k_j\,k^j - \frac{i}{k_0}\,P_{0j}\,k^j\right], \end{aligned} \tag{33.14}$$

and note from Einstein's equation (linearized), we still have $k^\mu\,k_\mu = 0 \rightarrow k_j\,k^j = k_0^2$ and from the original gauge choice, $P_{\mu\nu}\,k^\nu = 0 \rightarrow P_{0j}\,k^j = P_{00}\,k^0$, so that we can simplify the second line above:

$$\begin{aligned} h''^{\mu}_{\mu} &= -P^{\mu}_{\mu} - 2\,i\left[i\left(P_{00} + \frac{1}{2}\,P^{\alpha}_{\alpha}\right) - i\,P_{00}\right] \\ &= 0. \end{aligned} \tag{33.15}$$

We see that the metric perturbation is also "traceless". This "transverse traceless" gauge (finally) is completely fixed, with two independent degrees of freedom left. Because the perturbation is traceless, we have $\bar{h}_{\mu\nu} = h_{\mu\nu}$ without ambiguity, and we are ready to do some physics.

33.2 E&M

33.2.1 Gauge

Remember how this argument runs in standard electrodynamics: you start with the field equations for $\mathbf{E}$ and $\mathbf{B}$ (source-free):

$$\begin{aligned} \nabla \cdot \mathbf{E} = 0 \quad & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \quad & \nabla \times \mathbf{B} = \frac{\partial E}{\partial t} \end{aligned} \tag{33.16}$$

and introduced the potentials in the usual way: $\nabla \cdot \mathbf{B} = 0 \rightarrow \mathbf{B} = \nabla \times \mathbf{A}$ and then we observe that the curl of $\mathbf{E}$, while not (by itself) zero, can be augmented:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) \longrightarrow \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{33.17}$$

so we introduce a potential A_0 here, $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla A_0$, and we have automatically satisfied the above.

Now there is only the "source" equations (for now, zero):

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \longrightarrow \nabla^2 A_0 = -\frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) \tag{33.18}$$

and the vector one:

$$\left(\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) = \nabla \left(\nabla \cdot \mathbf{A} + \frac{\partial V}{\partial t} \right). \tag{33.19}$$

The usual observation is: 1. Adding the gradient of a scalar to $\mathbf{A}$ doesn't change $\nabla \times \mathbf{B}$, and 2. Adding the time derivative of that scalar to A_0 doesn't change $\mathbf{E}$. So we introduce

$$\mathbf{A}' = \mathbf{A} + \nabla \phi \quad A_0' = A_0 - \frac{\partial \phi}{\partial t} \tag{33.20}$$

which guarantees that $\mathbf{B}' = \mathbf{B}$.

So the $\mathbf{E}$, $\mathbf{B}$ content of the new potential A_0' and $\mathbf{A}'$ is identical to A_0 and $\mathbf{A}$, but now we have freedom to *choose* properties for the potentials.

In particular, we can impose Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, that simplifies the right-hand side of (33.18), giving us Laplace's equation. But then we're stuck with a tricky vector-potential equation. Of course, viewing these as components of a four-potential makes the situation somewhat clearer – in particular, both the scalar potential and the components of the vector potential satisfy wave equations – throwing out the time portion as we do in Coulomb gauge is not in the spirit of relativity.

So we choose Lorentz gauge which is $\partial^\mu A_\mu = 0$ with A_μ the usual four-vector. Then, as with our GR wave equation, we have:

$$\Box^2 A_\mu = 0. \tag{33.21}$$

33.2.2 Source Free E&M

Let's review the E&M derivation of light, just to remind ourselves of the way in which polarization crops up there. We start with the wave equation for the four-potential A_μ – already in Lorentz gauge (or we would not *have* a wave equation):

$$\left(-\partial_t^2 + \nabla^2\right) A_\mu = 0. \tag{33.22}$$

The idea is to introduce a Fourier transform solution – i.e. make the assumption $A_\mu(x) = P_\mu \, e^{i\,(\mathbf{k}\cdot\mathbf{x} - \omega\, t)}$ – this becomes, under the wave equation:

$$\left(\omega^2 - k^2\right) A_\mu = 0 \tag{33.23}$$

which tells us that the wave-four-vector defined as $k_\mu = (\omega, k_x, k_y, k_z)$ has $k^\mu k_\mu = 0$. In addition, we must satisfy the Lorentz gauge condition:

$$\partial^\mu A_\mu = 0 \longrightarrow P_0\,\omega + \mathbf{k}\cdot\mathbf{P} = 0 \tag{33.24}$$

with $P_\mu = (P_0, \mathbf{P})$. In other words, we have $k^\mu P_\mu = 0$. This gives us a set of algebraic relations.

For concreteness, consider a wave in the z direction with frequency ω. Demanding that $k_\mu k^\mu = 0$ gives $\omega = k_z$, and then the orthogonality with P_μ tells us that $-P_0 + P_z = 0$. This leaves us with two free components,

P_x and P_y (think of the P_z dependence of the electric and magnetic fields) in the polarization vector, just as we expect for light (i.e. the polarization vector is orthogonal to the propagation vector).

We just *set* P_0 to zero above – what is our justification for this move? There is un-fixed gauge freedom left in the potential A_μ, even once it is in divergenceless form. Think of a transformation $A'_\mu = A_\mu + \psi_{,\mu}$ for scalar ψ with $\partial^\mu A_\mu = 0$ already. We know this will leave us in Lorentz gauge provided:

$$\partial^\mu A'_\mu = \partial^\mu A_\mu + \partial^\mu \partial_\mu \psi = 0 \longrightarrow \Box^2 \psi = 0 \tag{33.25}$$

so we can make an *additional* gauge transformation, even when we are in Lorentz gauge. Take $\psi = Q\, e^{i\, k_\gamma\, x^\gamma}$, so that the D'Alembertian is zero – then

$$A'_\mu = A_\mu + i k_\mu Q e^{i\, k_\gamma\, x^\gamma}, \tag{33.26}$$

and we can use this to set, for example, $A'_0 = 0$ by appropriate choice of Q.

33.3 Source Free Linearized Gravity

We know already that we can set $h_{\mu\nu}$ to have zero components in $h_{0\nu} = h_{\nu 0} = 0$ directions, and this suggests that we just take the spatial part for our "real" perturbation. We will introduce Cartesian coordinates – a global inertial frame, that is after all, what we are doing in the linearization process. Let's be explicit, suppose we want to describe a wave with frequency ω traveling in the z direction according to our laboratory. Our wave equation, coupled to Einstein's equation and our gauge choice require us to solve the following:

$$\boxed{\begin{aligned} h_{\mu\nu} &= P_{\mu\nu}\, e^{i\, k_\mu\, x^\mu} \\ 0 &= P_{\mu\nu}\, k^\mu \\ 0 &= k_\mu\, k^\mu \\ 0 &= P_{0\nu} = P^\alpha_\alpha \end{aligned}} \tag{33.27}$$

setting $k_\mu \doteq (\omega, 0, 0, k_z)$ (the wavevector appropriate to z propagation with frequency ω), we have from $k_\mu\, k^\mu = 0$, $k_z = \pm\omega$. The upshot of the rest of the constraints is a final polarization tensor that looks like:

$$P_{\mu\nu} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -P_{yy} & P_{xy} & 0 \\ 0 & P_{xy} & P_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33.28}$$

Well that's great, but what *is* it? We are in a good position for interpretation, equivalent to the E&M case: A flat background with a field on top of it. We have the polarization tensor, we know the wavevector. The question, as always, is: what happens to particles? We need to go back to our test particle case and see how things move.

Referring back to our discussions of geodesic deviation, we consider two particles separated by s^γ (to avoid using η^γ as the separation vector). We will assume that the (spatial) velocity of the particles is small compared to the speed of light, so that we can write $\dot{x}^\beta \doteq (1,0,0,0)$, and then the "acceleration" becomes:

$$\frac{D^2}{D\tau^2}\, s^\alpha = -R^\alpha{}_{0\gamma 0}\, \dot{x}^0\, \dot{x}^0\, s^\gamma \tag{33.29}$$

leaving us with the task of computing the Riemann tensor (to first order) for $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}$ with our wave perturbation.

Let's take the right-hand side first – we discussed the linearized Riemann tensor when we originally constructed the linearized Einstein equation, it looks like:

$$R^\alpha{}_{\beta\gamma\delta} = \frac{1}{2}\,\epsilon\,\eta^{\alpha\rho}\left(h_{\beta\gamma,\delta\rho} + h_{\delta\rho,\beta\gamma} - h_{\beta\delta,\gamma\rho} - h_{\gamma\rho,\beta\delta}\right). \tag{33.30}$$

The relevant components (in our transverse traceless gauge) are

$$\begin{aligned} R^\alpha{}_{0\gamma 0} &= \frac{1}{2}\,\epsilon\,\eta^{\alpha\rho}\left(h_{0\gamma,0\rho} + h_{0\rho,0\gamma} - h_{00,\gamma\rho} - h_{\gamma\rho,00}\right) \\ &= -\frac{1}{2}\,\epsilon\,\eta^{\alpha\rho}\, h_{\gamma\rho,00} \end{aligned} \tag{33.31}$$

which isn't too bad. On the left, we have:

$$\frac{D^2}{D\tau^2}\, s^\gamma = \dot{x}^\alpha\, \dot{x}^\beta\, s^\gamma{}_{;\alpha\beta} + s^\gamma{}_{;\alpha}\left(\dot{x}^\alpha{}_{;\beta}\, \dot{x}^\beta\right) \tag{33.32}$$

the second term is zero because x^α is a geodesic by definition. Again, since we are working in the linearized theory, there is no difference to this order between covariant and normal derivatives, and in addition, we can convert $\tau \to t$ coordinate time, putting the left and right together and remembering that there are only spatial components to the metric:

$$\ddot{s}^i = \frac{1}{2}\,\epsilon\,\ddot{h}^i_j\,s^j. \tag{33.33}$$

One last linearization issue – let $s^i = \bar{s}^i + \epsilon\,\tilde{s}^i$, so that we can match orders in the expansion of the deviation and the Riemann tensor. Then we have two equations, one for order ϵ^0, one for ϵ^1 (and then the rest are $O(\epsilon^2)$):

$$\begin{aligned} \epsilon^0 &: \ddot{\bar{s}}^i = 0 \\ \epsilon^1 &: \ddot{\tilde{s}}^i = \frac{1}{2}\,\bar{s}^j\,\ddot{h}^i_j. \end{aligned} \tag{33.34}$$

The zeroth order equation tells us that $\bar{s}^i = \alpha^i\,t + \beta^i$, but let's agree to an initial condition: the masses are at rest at time $t = 0$ (before the wave arrives). Then we set $\alpha^i = 0$, and the second equation can be integrated (the remaining integration constant(s) β^i are just the initial separation):

$$s^i(t) = \beta^i + \frac{1}{2}\,\epsilon\,\beta^j\,h^i_j. \tag{33.35}$$

So far we have only used the gauge condition for the perturbation, which also implies that only the x and y components of s^i are non-zero. Let's put in our plane wave solution, with wave-vector pointing in the z direction, as shown in Figure 33.1.

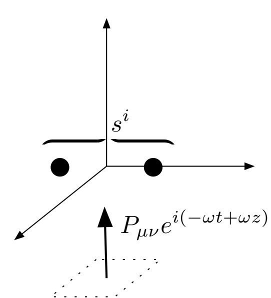

Figure 33.1: A gravitational wave moving in the $\hat{k}$ direction.

Using the form of the polarization tensor from (33.28), the separation vector is given by

$$\boxed{\begin{aligned}\mathbf{s} = &\left(s^x(0) + e^{i\,\omega\,(s^z(0)-t)}(s^y(0)\,P_{xy} - s^x(0)\,P_{yy})\right)\,\hat{\mathbf{x}} \\ &+\left(s^y(0) + e^{i\,\omega\,(s^z(0)-t)}\,(s^x(0)\,P_{xy} + s^y(0)\,P_{yy})\right)\,\hat{\mathbf{y}}.\end{aligned}} \tag{33.36}$$

To see the motion clearly, we set a pair of test masses on the x axis, and a pair on the y axis. With both pairs, we set $s^z(0) = 0$, the wave has no effect on that direction. In addition, it is useful to separate the motion into the two polarizations, obtained by setting $P_{xy} = 0$ and $P_{yy} = 0$. In Figure 33.2, we show the displacements for two pairs of test particles: one pair initially positioned on the x-axis, one pair on the y. By looking at the two polarizations separately, we can see the motivation for the names: $P_{yy} = P_+$, the "plus" polarization, and $P_{xy} = P_\times$, the "cross" polarization.

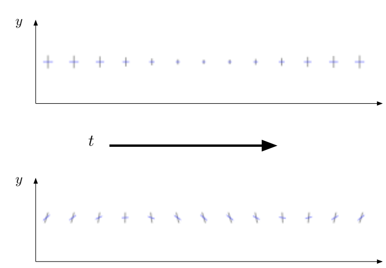

Figure 33.2: The "plus" polarization (top, $P_{xy} = 0$) shows the distance for initial $\hat{\mathbf{x}}$ (black) and $\hat{\mathbf{y}}$ (blue) directed separation vectors, the "cross" polarization (bottom, $P_{yy} = 0$) is shown for the same initial displacements.