Sourced Radiation

Lecture 34

Physics 411
Classical Mechanics II

November 21st, 2007

We are ready to move on to the source side of linearized waves. The point of this whole section has been to show the similarity between gravitational waves and the familiar E&M waves, just an extra index, but it does lead to some interesting physics. We have seen that if you want to detect gravitational waves, you need to measure a separation, and indeed, this is fundamentally how current detectors work – there are two test masses (mirrors) separated by a few kilometers, in the shape of an L, and you have effectively a giant interferometer, lasers are shot at the mirrors, you cancel them perfectly at the output and wait for a lack of cancellation, indicating that the mirrors have moved. Sounds easy enough. The problem, as we shall see, is twofold: 1. Gravitational waves are incredibly weak, both because of the coupling, and the quadrupole nature of the radiation, and 2. The frequency range of interest is on the order of one hertz – trucks, for example, are an issue.

34.1 Sources

We begin by returning to the wave equation, this time leaving the stress-tensor intact on the right-hand side. Let's re-state the problem here to set the notation:

$$\boxed{\begin{aligned} \partial_\rho \partial^\rho \, h_{\mu\nu} &= -16\,\pi\,T_{\mu\nu} \\ \partial^\rho \, h_{\nu\rho} &= 0. \end{aligned}} \tag{34.1}$$

Start with the top equation, this is a job for the Green's function. The gauge condition amounts to stress-tensor conservation.

34.1.1 Aside: Green's Functions

Here we will develop the Green's function for the D'Alembertian, the solution to $\partial^\rho \partial_\rho \phi = -\delta$. From this, we can build up solutions to the wave equation that are driven by sources, clearly the next step given our linearized left-hand side.

To a certain extent, classical physics, aside from interpretation, is the study of ODEs and PDEs. Most physical laws are expressed as derivative operators on some object of interest – trajectory coordinates, fields, potentials, etc. We have seen this in our discussion of general relativity over and over – Einstein's equation is an example, the linearized Einstein equation another, the equations of motion themselves are an example. Particularly for fields, though, the main tool is the equation of motion derived from an action – these will be, unless there are simplifying assumptions, partial derivative operators in all four dimensions.

With each PDE operator (and boundary conditions) comes an associated "Green's function" – the question is, how do we generically solve a PDE like:

$$\partial^\alpha \partial_\alpha \phi = -\rho \tag{34.2}$$

if someone hands us a source function ρ? The somewhat counterintuitive answer is: Solve the problem

$$\partial^\alpha \partial_\alpha G = -\delta(x^\mu - x'^\mu), \tag{34.3}$$

and then the solution to (34.2)

$$\phi(x^\mu) = \int d\tau' \, \rho(x'^\mu) \, G(x^\mu, x'^\mu). \tag{34.4}$$

That's a bunch of symbols, let me motivate with the three-dimensional Laplacian. We have

$$\nabla^2 \phi = -\rho, \tag{34.5}$$

which means, according to our prescription, we are interested in

$$\nabla^2 G = -\delta(\mathbf{x} - \mathbf{x}'), \tag{34.6}$$

and you should think of G as a "real" potential for a point source located at $\mathbf{x}'$ as measured at $\mathbf{x}$ (that's what the "source" $\rho = \delta(\mathbf{x} - \mathbf{x}')$ tells us). In forming the integral $\phi = \int d\tau \, \rho \, G$, we are taking the real distribution of charge (say) and viewing it as a set of points, each point contributes G to the potential and so the total potential is just the sum (integral) of all the points with the appropriate potential – a statement about superposition. To prove that this ϕ actually solves Poisson's equation, we use "Green's theorem"

$$\int \left(\phi \, \nabla^2 \, G - G \, \nabla^2 \, \phi\right) \, d\tau' = \int \left(\phi \, \nabla \, G - G \, \nabla \, \phi\right) \cdot d\mathbf{a}' \tag{34.7}$$

where the primes tell us the integration is over $\mathbf{x}'$. The integration occurs over whatever space we like, although it should include the $\mathbf{x}'$ of the distribution itself if we want to get an interesting answer – and under usual assumptions about the fields (that they fall off faster than a sphere), we take the surface at infinity, this just kills the right-hand side of the above, and then we can evaluate the left because it already involves the Laplacian:

$$\int d\tau' \left(-\phi \, \delta(\mathbf{x} - \mathbf{x}') + G \, \rho\right) = 0 \rightarrow -\phi(\mathbf{x}) + \int d\tau' \, G \, \rho = 0 \tag{34.8}$$

and we're done.

There are a couple of important facts about Green's functions that one should keep in mind: 1. They depend on a given differential operator *and* boundary conditions – the whole of the specified problem (we ignored that above), 2. They depend on the dimension of space (or space-time), 3. They rely on superposition – that is, they only apply to *linear* (or linearized) PDEs. A more useful (and relatively generic) property of Green's functions is their symmetry, $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$ – and indeed, for our problems, which have been specified only with the implicit boundary condition at $|\mathbf{x}| \rightarrow \infty$ (G vanishes there), one can go even further: $G(\mathbf{x}, \mathbf{x}') = G(|\mathbf{x} - \mathbf{x}'|)$.

The Green's function approach, while it looks like a cureall, requires us to compute the appropriate G for our problem. Because of the δ-function on the right-hand side, this is not trivial, and the calculation of these functions is sometimes tricky. For example, one way to approach the Laplacian solution is to specialize to spherical coordinates, and set the source coordinates $\mathbf{x}'$ at the origin – then the Green's function depends only on r, and away from the origin, we have:

$$G'' + \frac{2\,G'}{r} = 0 \rightarrow G(r) = \frac{\alpha}{r} + \beta. \tag{34.9}$$

The ODE *at* the origin has an infinite right-hand side – to get rid of this, we can integrate around an infinitesimal ball enclosing $r = 0$

$$-\int \delta(\mathbf{x})\, r^2\, \sin\theta\, d\theta\, d\phi\, dr = -1, \tag{34.10}$$

a fundamental property of the δ function – for the left-hand side, we have:

$$\int d\tau\, \nabla^2\, G = \int d\tau\, \nabla \cdot (\nabla\, G) = \int \nabla\, G \cdot \hat{\mathbf{n}}\, da = -4\,\pi\,\alpha = -1 \tag{34.11}$$

so we set the normalization to $\alpha = \frac{1}{4\pi}$ – the above says nothing about β, but in E&M at least, we know that β just sets the zero of the potential at some point. Our final solution is:

$$G(r) = \frac{1}{4\,\pi\,r} \tag{34.12}$$

and if we had the source position at $\mathbf{x}'$ rather than the origin,

$$\nabla^2\, G = -\delta(\mathbf{x} - \mathbf{x}') \rightarrow G(|\mathbf{x} - \mathbf{x}'|) = \frac{1}{4\,\pi\,\mathscr{r}} \tag{34.13}$$

with $\boldsymbol{\mathscr{r}} \equiv \mathbf{x} - \mathbf{x}'$.

The Green's function approach is useful, because in the linearized Einstein equations with source, we have:

$$\partial^\alpha \partial_\alpha\, h_{\mu\nu} = -16\,\pi\, T_{\mu\nu}. \tag{34.14}$$

So we ask the "point-source" question, what is the solution to

$$\partial^\alpha \partial_\alpha\, Q_{\mu\nu} = -\delta(x^\mu - x'^\mu) \tag{34.15}$$

(I'm using $Q_{\mu\nu}$ as the Green's function rather than $G_{\mu\nu}$). Notice that, when we expand out the flat space D'Alembertian, we get $-\partial_t^2 + \nabla^2$, so this is almost a Laplacian, and we can make it even more like a Laplacian by a temporal Fourier transform – multiply by $e^{i\,\omega\,t}$ and integrate over t:

$$
\begin{aligned}
-\int_{-\infty}^{\infty} dt\,(\partial_t^2\,Q_{\mu\nu}(t,\mathbf{x},\mathbf{x}'))\,e^{i\,\omega\,t}+\nabla^2\int_{-\infty}^{\infty} dt\,Q_{\mu\nu}(t,\mathbf{x},\mathbf{x}')\,e^{i\,\omega\,t}&=-\delta(\mathbf{x}-\mathbf{x}')\,e^{i\,\omega\,t'}\\
-\int_{-\infty}^{\infty} dt\,(-\omega^2)\,Q_{\mu\nu}(t,\mathbf{x},\mathbf{x}')\,e^{i\,\omega\,t}+\nabla^2\int_{-\infty}^{\infty} dt\,Q_{\mu\nu}(t,\mathbf{x},\mathbf{x}')\,e^{i\,\omega\,t}&=-\delta(\mathbf{x}-\mathbf{x}')\,e^{i\,\omega\,t'}
\end{aligned}
\tag{34.16}
$$

where we have integrated-by-parts twice to get the second line. Now to set the Fourier transform normalization, define

$$
\begin{aligned}
\tilde{Q}_{\mu\nu}(\omega,\mathbf{x},\mathbf{x}')&=\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} dt\,Q_{\mu\nu}(t,\mathbf{x},\mathbf{x}')\,e^{i\,\omega\,t}\\
Q_{\mu\nu}(t,\mathbf{x},\mathbf{x}')&=\frac{1}{\sqrt{2\,\pi}}\int_{-\infty}^{\infty} d\omega\,\tilde{Q}_{\mu\nu}(\omega,\mathbf{x},\mathbf{x}')\,e^{-i\,\omega\,t}
\end{aligned}
\tag{34.17}
$$

and our D'Alembertian becomes

$$
(\omega^2+\nabla^2)\,\tilde{Q}_{\mu\nu}(\omega,\mathbf{x},\mathbf{x}')=-\frac{1}{\sqrt{2\,\pi}}\,\delta(\mathbf{x}-\mathbf{x}')\,e^{i\,\omega\,t'}. \tag{34.18}
$$

The differential operator on the left-hand side of this is called the "Helmholtz" operator, and its Greens function is known, but let's continue in our derivation. We know, because this PDE comes to us with the implicit condition that $h_{\mu\nu}\rightarrow 0$ at infinity, that the Green's function depends only on the magnitude of the difference: $\mathbf{x}-\mathbf{x}'$ between the (flat-space) observation point $\mathbf{x}$ and a source point $\mathbf{x}'$. Then the spherical Laplacian may be used as usual for $\nabla^2\tilde{Q}_{\mu\nu}(\omega,\mathscr{r}\equiv|\mathbf{x}-\mathbf{x}'|)$,

$$
\left(\omega^2+\frac{d^2}{d\mathscr{r}^2}\right)\left[e^{-i\,\omega\,t'}\,\mathscr{r}\,\tilde{Q}_{\mu\nu}(\omega,\mathscr{r})\right]=-\frac{\mathscr{r}}{\sqrt{2\,\pi}}\,\delta(\boldsymbol{\mathscr{r}}). \tag{34.19}
$$

This equation is only valid away from $\mathscr{r}=0$, since we've multiplied through by $\mathscr{r}$ to get it in this form – but then we can solve the homogenous part immediately, it's just a harmonic oscillator

$$
e^{-i\,\omega\,t'}\,\mathscr{r}\,\tilde{Q}_{\mu\nu}(\omega,\mathscr{r})=\alpha\,e^{i\,\omega\,\mathscr{r}}+\beta\,e^{-i\,\omega\,\mathscr{r}}\;\rightarrow\tilde{Q}_{\mu\nu}(\omega,\mathscr{r})=\frac{e^{i\,\omega\,t'}}{\mathscr{r}}\left(\alpha\,e^{i\,\omega\,\mathscr{r}}+\beta\,e^{-i\,\omega\,\mathscr{r}}\right). \tag{34.20}
$$

We need to check the spatial part – does this reduce to a δ function, and what normalization should we use? The quickest way to do this is to look at the limit of $Q_{\mu\nu}(\omega,\mathscr{r})$ as $\mathscr{r}\rightarrow 0$

$$
\lim_{\mathscr{r}\rightarrow 0}\tilde{Q}_{\mu\nu}(\omega,\mathscr{r})=\frac{\alpha}{\mathscr{r}}+\alpha\,(i\,\omega)+\frac{\beta}{\mathscr{r}}-\beta\,(i\,\omega)+O(\mathscr{r}) \tag{34.21}
$$

so that the behavior of this function near the origin is identical to the three-dimensional Green's function for the Laplacian (34.13), modulo constants. Then we should set, $\alpha + \beta = \frac{1}{\sqrt{2\,\pi}\,4\,\pi}$, and now we can write the final form for the transformed Green's function:

$$\boxed{\tilde{Q}_{\mu\nu}(\omega, \mathscr{r}\,) = \frac{1}{4\,\pi\,\sqrt{2\,\pi}\,\mathscr{r}}\,e^{i\,\omega\,(t' \pm \mathscr{r}\,)},} \tag{34.22}$$

where I've written this as two separate solutions rather than a linear combination. Now we Fourier transform back to t

$$\begin{aligned} Q_{\mu\nu}(t, \mathscr{r}\,) &= \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\,\omega\,t} \left(\frac{1}{4\,\pi\,\sqrt{2\,\pi}\,\mathscr{r}}\,e^{i\,\omega\,(t' \pm \mathscr{r}\,)} \right) \\ &= \frac{1}{4\,\pi\,(2\,\pi)\,\mathscr{r}} \int_{-\infty}^{\infty} d\omega\, e^{i\,\omega\,(t' - t \pm \mathscr{r}\,)}, \end{aligned} \tag{34.23}$$

and note that

$$\frac{1}{2\,\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\,\omega\,(t-p)} = \delta(t-p) \tag{34.24}$$

so

$$\boxed{Q_{\mu\nu}(t, \mathscr{r}\,) = \frac{1}{4\,\pi\,\mathscr{r}}\,\delta\big(t - t' \pm \mathscr{r}\,\big) = \frac{1}{4\,\pi\,\mathscr{r}}\,\delta\big(t' - t \mp \mathscr{r}\,\big)} \tag{34.25}$$

is the appropriate Green's function in terms of time.

34.2 Source Approximations and Manipulation

We have not made any assumptions about the sources themselves yet – and because of the "spatial dependence on time" (through $t_- = t - \mathscr{r}\,$), we need to be careful to separate out all the spatial dependence before making "far away" types of approximation. The easiest way to disentangle is to Fourier transform both sides of

$$\Box^2\,h_{\mu\nu} = -16\,\pi\,T_{\mu\nu} \longrightarrow h_{\mu\nu} = 4 \int \frac{T_{\mu\nu}(t', \mathbf{x}')}{\mathscr{r}}\,\delta(t' - t \mp \mathscr{r}\,)\,d\tau'\,dt' \tag{34.26}$$

where $d\tau' = dx'\,dy'\,dz'$ (just the spatial portion of the integration).

Let's focus on the right-hand side first – multiply by $e^{i\,\omega\,t}$ and integrate over t, using the δ to perform the dt' integral, we have:

$$4 \int d\tau'\,dt\,e^{i\,\omega\,(t - \mathscr{r}\,)}\,\frac{e^{i\,\omega\,\mathscr{r}}}{\mathscr{r}}\,T_{\mu\nu}(t, \mathbf{x}'), \tag{34.27}$$

where we multiply and divide by the spatial term so that we can define the retarded time $t_- \equiv t - \text{ᴤ}$. A change of variables gives:

$$
\begin{aligned}
&= 4 \int d\tau' \, dt_- \, \frac{e^{i\,\omega\,\text{ᴤ}}}{\text{ᴤ}} \, e^{i\,\omega\,t_-} \, T_{\mu\nu}(t_-, \mathbf{x}') \\
&= 4\sqrt{2\,\pi} \int d\tau' \, \frac{e^{i\,\omega\,\text{ᴤ}}}{\text{ᴤ}} \, \tilde{T}_{\mu\nu}(\omega, \mathbf{x}').
\end{aligned}
\tag{34.28}
$$

We make the usual approximation, that $\text{ᴤ} = |\mathbf{x} - \mathbf{x}'|$ is dominated by the observation distance $|\mathbf{x}|$, i.e. we are far away and the origin is centered inside the mass distribution. Then approximately, we have:

$$
4\sqrt{2\,\pi} \int d\tau' \, \frac{e^{i\,\omega\,\text{ᴤ}}}{\text{ᴤ}} \, \tilde{T}_{\mu\nu}(\omega, \mathbf{x}') \approx 4\sqrt{2\,\pi} \, \frac{e^{i\,\omega\,|\mathbf{x}|}}{|\mathbf{x}|} \int d\tau' \, \tilde{T}_{\mu\nu}(\omega, \mathbf{x}'). \tag{34.29}
$$

Now we can attack the Fourier transform of the source term. Writing the full (linearized) Einstein equation for distant sources,

$$
\boxed{\tilde{h}^{\mu\nu}(\omega, \mathbf{x}) \approx 4 \, \frac{e^{i\,\omega\,|\mathbf{x}|}}{|\mathbf{x}|} \int d\tau' \, \tilde{T}^{\mu\nu}(\omega, \mathbf{x}'),} \tag{34.30}
$$

the first thing to bear in mind is that we need only deal with the spatial components on either side – this comes from the gauge condition $\partial^\mu h_{\mu\nu} = 0$, and the equivalent statement for the stress tensor (linearized conservation):

$$
\begin{aligned}
\partial_\mu \, h^{\mu\nu} &= 0 \rightarrow (-i\,\omega) \, \tilde{h}^{0\nu} + \partial_j \, \tilde{h}^{j\nu} = 0 \\
\partial_\mu \, T^{\mu\nu} &= 0 \rightarrow (-i\,\omega) \, \tilde{T}^{0\nu} + \partial_j \, \tilde{T}^{j\nu} = 0.
\end{aligned}
\tag{34.31}
$$

The condition allows us to solve for the space-space components in terms of the others. So referring only to (i, j) spatial indices, we can rewrite the integrand on the right as a total divergence and a remaining piece:

$$
\int d\tau' \, \tilde{T}^{jk} = \int d\tau' \left(\partial_s' \, (\tilde{T}^{sk} \, x'^j) - x'^j \, \partial_s' \, T^{sk} \right), \tag{34.32}
$$

where the total divergence is turned into a surface integral, and as long as the distribution is not infinite, can be set to zero (extend the volume of integration to infinity, T^{jk} will be zero outside of some finite region, then the source at infinity is zero). Replacing the spatial derivatives with the momenta (using conservation for Fourier modes, as shown above) and noting that h^{jk} is symmetric in (j, k), we can rewrite and repeat the process:

$$\begin{aligned}\int d\tau' \, \tilde{T}^{jk} &= -\frac{i\,\omega}{2} \int d\tau' \left(x'^k \, \tilde{T}^{0j} + x'^j \, \tilde{T}^{0k} \right) \\ &= -\frac{i\,\omega}{2} \int d\tau' \left(\partial'_s \left(x'^k \, x'^j \, \tilde{T}^{0s} \right) - x'^k \, x'^j \, \partial'_s \, \tilde{T}^{0s} \right) \\ &= \frac{(i\,\omega)^2}{2} \int d\tau' \, x'^k \, x'^j \, \tilde{T}^{00}.\end{aligned} \tag{34.33}$$

The above is just what we would call the (Fourier transform of) an integral related to the quadrupole moment of the source in E&M. Again, we see the importance of a second index – in E&M, all of this machinery is applied to the wave equation for the potential $\partial^\alpha \partial_\alpha \, A^\mu = -j^\mu$. The move from two spatial indices to one via a total derivative and the conservation law for $T^{\mu\nu}$ is the mathematical statement of charge conservation (energy density conservation here). That is identical to E&M, and there, forces us to look at dipole radiation as the first radiating term. With the move from one spatial index to none, as above, we are effectively using momentum conservation in GR, so there is no dipole radiation, in addition to no monopole, and we are left with the quadrupole as the first contributing term.

We have, on the Fourier side, the relation:

$$\tilde{h}^{jk}(\omega, \mathbf{x}) = 4 \, \frac{e^{i\,\omega\,|\mathbf{x}|}}{|\mathbf{x}|} \, \frac{(i\,\omega)^2}{2} \int d\tau' \, x'^j \, x'^k \, \tilde{T}^{00}(\omega, \mathbf{x}'). \tag{34.34}$$

Multiplying by $e^{-i\,\omega\,t}$ and integrating, we can return to time,

$$\begin{aligned}h^{jk}(t, \mathbf{x}) &= \frac{2}{|\mathbf{x}|} \, \frac{d}{dt^2} \int d\omega \, e^{-i\,\omega\,(t - |\mathbf{x}|)} \int d\tau' \, x'^j \, x'^k \, \tilde{T}^{00}(\omega, \mathbf{x}') \\ &= \frac{2}{|\mathbf{x}|} \, \frac{d^2}{dt^2} \left(\int d\tau \, x'^j \, x'^k \, T^{00}(t, \mathbf{x}') \right) \bigg|_{t = t_- \equiv t - |\mathbf{x}|}.\end{aligned} \tag{34.35}$$

Defining the energy density quadrupole to be three times the quantity in parenthesis (a normalization),

$$\boxed{\begin{aligned}h^{jk}(t, \mathbf{x}) &= \frac{2}{3\,|\mathbf{x}|} \, \ddot{q}^{jk}|_{t=t_-} \\ q^{jk} &\equiv 3 \int d\tau' \, T^{00}(t, \mathbf{x}') \, x'^j \, x'^k.\end{aligned}} \tag{34.36}$$

34.3 Example – Circular Orbits

Let's take a simple example to get a general picture of the connection between metric perturbations and sources. Our model system is a small body in a circular orbit about a more massive one, as shown in Figure 34.1.

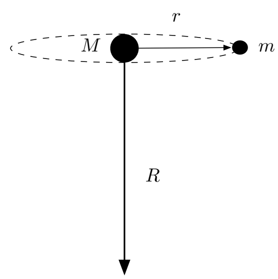

Figure 34.1: A circular orbit for a massive body M and a smaller body m observed on-axis from a distance R away.

Remember the Hamiltonian form of Newtonian orbits,

$$\begin{aligned} \dot{r}^2 &= \frac{2\,H}{r^2}\,(r - r_1)\,(r - r_1) \\ r_1 &= \frac{1}{2\,H}\,(-M + \sqrt{2\,H\,J_z^2 + M^2}) \\ r_2 &= \frac{1}{2\,H}\,(-M - \sqrt{2\,H\,J_z^2 + M^2}) \end{aligned} \tag{34.37}$$

where H is the Hamiltonian (the energy in this context), J_z is the angular momentum of the orbiting body and r_1, r_2 define the turning points of the motion. For a circular orbit, what we mean is that $r_1 = r_2$, i.e. the point of closest approach is identical to the point of furthest approach. This gives us a relation for the orbital energy H: $H = -\frac{M^2}{2\,J_z^2}$. In addition, we can *set* the $r(t)$ coordinate to have the value $r_1 = r_2 = \frac{J_z^2}{M}$ for all time – that's consistent. Then the right-hand side of the radial equation above is zero, as is the left. Now we can ask, what is the period of the motion? The answer is provided by $\phi(t)$, we integrate over one full period,

$$\dot{\phi} = \frac{J_z}{r^2} = \sqrt{\frac{M}{r^3}} \rightarrow \int_0^{2\pi} d\phi = \int_0^T dt\,\sqrt{\frac{M}{r^3}} \rightarrow T \quad = 2\,\pi\,\sqrt{\frac{r^3}{M}}, \tag{34.38}$$

from which we can calculate the angular frequency $\omega = \frac{2\pi}{T} = \sqrt{\frac{M}{r^3}}$. Then in Cartesian coordinates, we can write the trajectory as

$$\begin{aligned} x' &= r\,\cos(\omega\,t) \\ y' &= r\,\sin(\omega\,t) \\ z' &= 0. \end{aligned} \tag{34.39}$$

The energy density is just that of a point mass m evaluated along its trajectory:

$$T^{00} = m\,\delta^3(\mathbf{r}') = m\,\delta(z')\,\delta(x' - r\,\cos(\omega\,t))\,\delta(y' - r\,\sin(\omega\,t)), \tag{34.40}$$

and we can calculate the quadrupole directly

$$\begin{aligned} q^{jk} &= 3\int d\tau'\,T^{00}(x')\,x'^j\,x'^k \\ &\doteq 3\,m\int d\tau'\,\delta(z')\,\delta(x' - r\,\cos(\omega\,t))\,\delta(y' - r\,\sin(\omega\,t)) \begin{pmatrix} x'^2 & x'\,y' & x'\,z' \\ x'\,y' & y'^2 & y'\,z' \\ x'\,z' & y'\,z' & z'^2 \end{pmatrix} \\ &= 3\,m \begin{pmatrix} r^2\,\cos^2(\omega\,t) & r^2\,\sin(\omega\,t)\,\cos(\omega\,t) & 0 \\ r^2\,\sin(\omega\,t)\,\cos(\omega\,t) & r^2\,\sin^2(\omega\,t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{34.41}$$

Using (34.36), the spatial portion of the metric perturbation is

$$h^{jk}(t, \mathbf{x}) \doteq \frac{4\,r^2\,m\,\omega^2}{R} \begin{pmatrix} \cos(2\,\omega\,t_-) & \sin(2\,\omega\,t_-) & 0 \\ \sin(2\,\omega\,t_-) & -\cos(2\,\omega\,t_-) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{34.42}$$

For circular orbits, then, we can replace ω in the magnitude portion, the factor in front of the matrix is $\frac{4\,m\,M}{r\,R}$.

We have constructed the wave to be in the z-direction, and the form of the spatial portion tells us that the perturbation is transverse (already) and traceless. One of the hallmarks of gravitational waves, owing to the two time-derivatives of the quadrupole, is that oscillations occur with twice the frequency of the source (for circular orbits).

From our discussion of geodesic deviation of test masses, we know that the separation as a function of time will go roughly like h_{ij} itself times the initial separation ($s(t) \approx \beta h(t)$). Given the above form for h_{ij} derived from a circular orbit, we can get an idea of the sensitivity requirements for a gravitational wave detector.

Suppose we are viewing a solar-like orbit, with a massive body (the sun) and a much smaller body (the earth) in an approximately circular orbit. If we view from a platform located, say $R = 10\,r$ (ten times the distance to the sun), then we need to measure a separation given by

$$\Delta s(t) \sim \beta\, h \tag{34.43}$$

where β is the initial separation of the test masses, and h is

$$h \sim \frac{4\,m\,M}{r\,R} = \frac{2\,m\,M}{5\,r^2}. \tag{34.44}$$

If we put in $M \approx 1.5$ km, and the earth has $m \approx 4.4 \times 10^{-6}$ km, with distance between the two $r = 1$ AU $\approx 1.5 \times 10^8$ km. Then

$$h \sim \frac{2 \times (1.5 \text{ km}) \times \left(4.4 \times 10^{-6} \text{ km}\right)}{5 \times (1.5 \times 10^8 \text{ km})^2} \approx 1.2 \times 10^{-22}. \tag{34.45}$$

We take a reasonable initial separation for the test masses, say $\beta \sim 1$ km, then our instrument needs to be sensitive to

$$\Delta s \sim h\beta \sim \left(1.2 \times 10^{-22}\right) \times (1 \text{ km}) \approx 1.2 \times 10^{-22} \text{ km} \approx 1.2 \times 10^{-9} \text{Å}. \tag{34.46}$$

And one can compare this with the radius of the Hydrogen atom, approximately .5 Å.

There are better sources, of course, the earth-sun system is a very weak radiator, but while there are more massive, faster systems around, they also tend to be further away. The upshot is, we are pretty much stuck measuring (very) small separations. Amazingly, the ground-based detectors (LIGO, for example) are getting close to the sensitivity limit in some frequency ranges.

One last point about gravitational waves. It is not possible to define local energy in the gravitational field – this is just the usual statement about equivalence, how do you separate the background space-time from the portion that would carry energy? To put it another way: in general, we know that space-time is locally flat, that's the starting point of all of this. So in the local frame, where the metric and connection vanish in a small neighborhood, we would say there is no energy associated with the metric. One can define total energy at certain points (like infinity) but not local energy density.

That's surprising when we think of linearized gravitational radiation, which has a flat background and waves propagating on it just like E&M – why can't we mimic the definition of field energy there? We can, but this is not a self-consistent procedure in GR, the linearization effectively breaks down, so it's not correct. However, if one insists, and proceeds carefully, it is possible to talk about the energy radiated in gravitational waves from a source like the circular one we have been considering.

Energy loss for a Newtonian orbit implies a period shift, and in 1993, Hulse and Taylor won the Nobel prize for observing the shift associated with gravitational wave energy loss (observed over ≈ 20 years) in a binary system. This is the first indirect evidence for gravitational radiation. Current experiments like LIGO and the space-based LISA are attempting a direct observation through a separation distance measurement.