Area Minimization

Lecture 36

Physics 411
Classical Mechanics II

December 3rd, 2007

To describe a relativistic string, rather than a point particle, we need two parameters (a surface rather than a line). So we begin by studying the more familiar surface minimization form of an action – Euclidean minimal surfaces. This introduces the two-parameter variation that we can extend to Minkowski spacetime, developing the Nambu-Goto action as a natural extension of area minimization there.

36.1 Surfaces

For a point particle, the ultimate goal of an action formulation is a description of the equations of motion that will allow us to solve for $\mathbf{x}(t)$, the position of the particle as a function of time. In the relativistic development of the action, we saw that a more natural parameter was τ, the proper time of the particle. The action itself was reparametrization invariant, though, so we could go back and forth between the coordinate t and the proper time τ. In addition, we were led to the correct relativistic action by a minimization of path requirement (especially true in the curved GR setting).

Suppose we consider an extended object, that is, a surface with two parameters, defined by $\mathbf{x}(u, v)$. We would like to find equations of motion that allow us to solve for the surface – there is no *a priori* obvious action, and it is not even clear physically what we want out of a theory of surfaces. After all, there is no Newton's second law governing the time evolution of a surface, nor is there any obvious physics to be found in a free surface. So we proceed by analogy – if the point particle action is developed as a length (of trajectory) minimization, perhaps the surface action can be usefully thought of as an area minimization.

For a small change du and dv in the parameters, we span a parallelogram – this is obvious from the response of $\mathbf{x}(u+du,v)$ and $\mathbf{x}(u,v+dv)$:

$$d\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}\, du \qquad d\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}\, dv, \tag{36.1}$$

defining two vectors that share an origin. This is really just a gradient, so it is no surprise that we get these two vector contributions. The area (squared) spanned by these vectors is:

$$dA^2 = (d\mathbf{x}_u \times d\mathbf{x}_v) \cdot (d\mathbf{x}_u \times d\mathbf{x}_v). \tag{36.2}$$

If we play our usual indexing games, we can make the above clear:

$$\begin{aligned} dA^2 &= (\epsilon_{ijk}\, dx_u^j\, dx_v^k)\,(\epsilon_{i\ell m}\, dx_u^\ell\, dx_v^m) \\ &= (\delta_{j\ell}\,\delta_{km} - \delta_{jm}\,\delta_{k\ell})\, dx_u^j\, dx_v^k\, dx_u^\ell\, dx_v^m \\ &= (d\mathbf{x}_u \cdot d\mathbf{x}_u)\,(d\mathbf{x}_v \cdot d\mathbf{x}_v) - (d\mathbf{x}_u \cdot d\mathbf{x}_v)^2. \end{aligned} \tag{36.3}$$

By now, it should come as no surprise that this area element can be written as a determinant – consider the matrix:

$$g_{ij} \doteq \begin{pmatrix} \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v} \\ \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial v} \end{pmatrix}, \tag{36.4}$$

then we have

$$dA = \sqrt{g}\, du\, dv. \tag{36.5}$$

That's interesting, and g_{ij} here is called an induced metric (hence the suggestive letter), it tells us how areas transform under the map from (u,v) space to $\mathbf{x} \doteq (x(u,v), y(u,v), z(u,v))$. The matrix g_{ij} is certainly symmetric, and invertible (by assumption), so it is a candidate for a metric – the clincher: it transforms like a second rank tensor under $u \longrightarrow \bar{u}$, $v \longrightarrow \bar{v}$. Then we know, from previous considerations of integrands, that the action:

$$\boxed{S = \int dA = \int \sqrt{g}\, du\, dv} \tag{36.6}$$

is a scalar.

36.2 Surface Variation

Now we ask the usual question: What surface $\mathbf{x}(u,v)$ minimizes the area action? Let $\mathbf{X}' \equiv \frac{\partial \mathbf{x}}{\partial u}$ and $\dot{\mathbf{X}} \equiv \frac{\partial \mathbf{x}}{\partial v}$ (we use capital $\mathbf{X}$ to denote the surface solution rather than the coordinates $\mathbf{x}$). Then explicitly, we have

$$S = \int \sqrt{\dot{\mathbf{X}}^2\,\mathbf{X}'^2 - (\dot{\mathbf{X}}\cdot\mathbf{X}')^2}\,du\,dv. \tag{36.7}$$

The equations of motion are given, as always (note that there is no $\mathbf{X}$ dependence) by

$$\frac{\partial}{\partial u}\left(\frac{\partial \mathcal{L}}{\partial \mathbf{X}'}\right) + \frac{\partial}{\partial v}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{X}}}\right) = 0, \tag{36.8}$$

which is just

$$\frac{\partial}{\partial u}\left(\frac{X'^2\,\dot{\mathbf{X}} - (\mathbf{X}'\cdot\dot{\mathbf{X}})\,\mathbf{X}'}{\sqrt{\dot{\mathbf{X}}^2\,\mathbf{X}'^2 - (\dot{\mathbf{X}}\cdot\mathbf{X}')^2}}\right) + \frac{\partial}{\partial v}\left(\frac{\dot{X}^2\,\mathbf{X}' - (\mathbf{X}'\cdot\dot{\mathbf{X}})\,\dot{\mathbf{X}}}{\sqrt{\dot{\mathbf{X}}^2\,\mathbf{X}'^2 - (\dot{\mathbf{X}}\cdot\mathbf{X}')^2}}\right) = 0. \tag{36.9}$$

An unenlightening result if there ever was one – but we display it here to emphasize the difficulty, given a generic pair (u, v), in writing down, much less solving, the "equations of motion" for the surface-area minimizing solution.

The point, as we saw in the case of the relativistic point particle action, is that we can use reparametrization invariance *prior* to varying to simplify the result.

36.2.1 Reparametrization – Minimal Surface in Cylindrical Geometry

Suppose we want to find the area-minimizing two-dimensional surface that connects two rings as shown in Figure 36.1.

Using cylindrical coordinates for $\mathbf{X}$: $d\mathbf{X} \doteq (ds(u, v), d\phi(u, v), dz(u, v))$, the natural parametrization here is to set $u = \phi$, $v = z$. We can do this with impunity, they are just labels. The dot product takes its usual form for cylindrical coordinates: $d\ell^2 = ds^2 + s^2\,d\phi^2 + dz^2$, and this must be respected in the action. Finally, we do not expect the cylindrical radial coordinate to depend on ϕ – that's a symmetry ansatz we are adding, it should be supported by the solution to the full equations of motion (i.e. leaving in the full (u, v) dependence), and as usual, we must be careful about making this assumption before we vary. Forging ahead, we can write the action (noting that $\frac{\partial \phi(u,v)}{\partial u} = 1$, $\frac{\partial \phi(u,v)}{\partial v} = 0$, since $(u = \phi, v = z)$, and $\frac{\partial z(u,v)}{\partial u} = 0$,

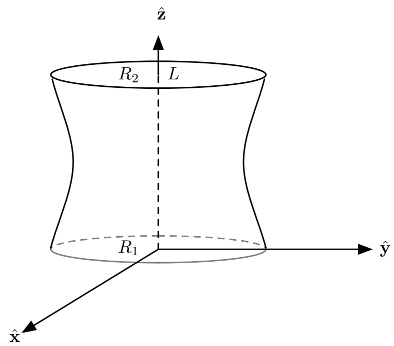

Figure 36.1: Two rings, of radius R_1 and R_2 sitting at $z = 0$ and $z = L$. We want to find the surface connecting the rings that minimizes the total area.

$\frac{\partial z(u,v)}{\partial v} = 1$ for the same reason), assuming $s(u, v) = s(v)$:

$$S = \int_0^{2\pi} \int_0^L \underbrace{s(z)\sqrt{1 + s'(z)^2}\, ds\, d\phi}_{\equiv \mathcal{L}}. \tag{36.10}$$

The variation is trivial, and tells us that:

$$\frac{d}{dz}\frac{\partial L}{\partial s'} - \frac{\partial L}{\partial s} = 0 = \frac{s\, s'' - s'^2 - 1}{(1 + s'^2)^{3/2}}, \tag{36.11}$$

the solution to which is

$$s(z) = \alpha \cosh\left(\frac{z - \beta}{\alpha}\right). \tag{36.12}$$

In order to set boundary conditions, we need to solve for α and β – this is not as easy as it appears. The $z = 0$ boundary condition gives

$$\alpha \cosh\left(\frac{\beta}{\alpha}\right) = R_1, \tag{36.13}$$

and must simultaneously solve

$$\alpha \cosh\left(\frac{L - \beta}{\alpha}\right) = R_2. \tag{36.14}$$

Is it always possible to satisy these two equations for arbitrary R_1, R_2 and L? If we set $R_1 = R_2 = 1$, we can solve the first equation for B, then

$$s(z) = A \cosh(z/A - \text{sech}^{-1}(A)). \tag{36.15}$$

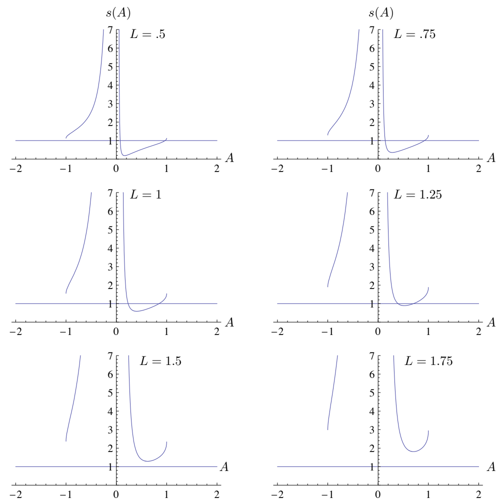

Figure 36.2: The function $s(z = L)$ for different values of A – we want A such that $s(L) = 1$ at some point. For $L > 1.25$ (approximately), no real A exists.

To find A, we need L, and we can see from Figure 36.2 that there does not necessarily exist an A for all separation lengths.

Physically, there is a critical separation distance – beyond that distance, the minimal surface is not described by the above solution, but is two separate sheets covering the rings, with a total area of $2(\pi R^2) = 2\pi$ for $R = 1$. For lengths L that have two solutions for A, we expect one to be less than 6π, one greater. Take $L = 1$, then numerically, the two values for A are $A_1 \sim .235095$ and $A_2 \sim .848338$. These lead to areas:

$$\int_0^{2\pi} \int_0^1 L\, d\phi\, dv \sim \left\{ \begin{array}{ll} 6.283 & A_1 \\ 5.992 & A_2 \end{array} \right. . \tag{36.16}$$

As expected, one of these is less than the flat solution, one greater.

36.3 Relativistic String

Very little changes structurally if we consider a string rather than a spatial surface. We are still minimizing an "area", but this time, our target space is not just three-dimensional Euclidean space with its natural dot product.

Instead, we are interested in Minkowski space-time, which also has a dot product. If we make the formal replacement:

$$\boxed{\mathbf{X} \cdot \mathbf{X} \longrightarrow X^\mu \, \eta_{\mu\nu} \, X^\nu,} \tag{36.17}$$

in (36.4) then almost everything we have done so far carries over without change. We can also relabel our parameters – these are typically called τ and σ rather than u and v, and this just reminds us to think of τ as a proper-time-like parameter (going from negative to positive infinity), and σ as a curve parameter that takes us along a string (so it is finite or closed).

The only other modification we must make to our Euclidean area action is to change the sign under the square root, this is a familiar shift, but comes about here in a more subtle way (remember that g_{ij} is an induced metric). We want a string to "be relativistic", but what does this mean for the derivatives of $X^\mu(\tau, \sigma)$ w.r.t. τ and σ?

Well, if we think of a point particle, parametrized by τ, then we know that the world-line of the point particle has, at any point, a time-like vector that points in the direction of increasing τ. This vector must be time-like, as it defines the local temporal axis for the particle at rest (and hence, near the point, any change $d\tau$ results in a change only in the temporal component of the particle's motion) as shown in Figure 36.3

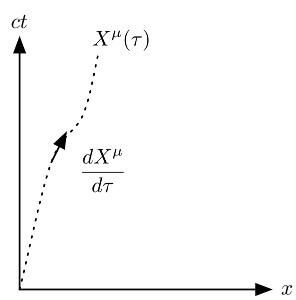

Figure 36.3: A particle moves along a world line as shown. At any point, the tangent to the curve is given by $\frac{dX^\mu}{d\tau}$, and this must be a time-like vector, defining, as it does, the purely temporal direction of motion.

When we think of a string, we imagine a world-sheet, a surface mapped out in the $(c\,t, x)$ plane above (and more generally, in the full four-dimensional space-time). Our first requirement is that points along the string should behave like point particles, i.e. there should exist, for any σ, a time-like direction. The space-like direction comes along the σ-parametrized direction of the curve, and just represents a tangent vector in that direction. We do not allow degenerate cases, i.e. we cannot have two space-like vectors, else a portion of the string is moving faster than light (or at the speed of light – in fact, the end points of a string can move at this speed), and two time-like directions implies that we have a point-particle with no spatial extent, a situation we already know something about.

Once we have two linearly independent directions, one timelike, one space-like, it is relatively easy to show that the quantity:

$$(\dot{\mathbf{X}} \cdot \dot{\mathbf{X}})\,(\mathbf{X}' \cdot \mathbf{X}') - (\dot{\mathbf{X}} \cdot \mathbf{X}')^2 < 0 \tag{36.18}$$

(with $\dot{\mathbf{X}} \doteq \frac{dX^\mu}{d\tau}$, $\mathbf{X}' \doteq \frac{dX^\mu}{d\sigma}$) and so to get a non-imaginary area, we need to put a minus sign under the square root in the action:

$$\boxed{S_{NG} \sim \int \sqrt{-g}\, d\tau\, d\sigma.} \tag{36.19}$$

Aside from normalization, this is the Nambu-Goto classical string action.