Relativistic String Solution

Lecture 37

Physics 411
Classical Mechanics II

December 5th, 2007

37.1 Nambu-Goto Variation

The Nambu-Goto action we found last time, now with some units in it, is

$$\boxed{S_{NG} = -\frac{T_0}{c}\int \sqrt{-g}\, d\sigma\, d\tau = -\frac{T_0}{c}\int \sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2\,(X')^2}\, d\sigma\, d\tau,} \tag{37.1}$$

with $\dot{X} \equiv \frac{\partial X^\mu}{\partial \tau}$, $X' \equiv \frac{\partial X^\mu}{\partial \sigma}$, and the dot product is understood in the Minkowski sense: $A \cdot B = A^\alpha\, B^\beta\, \eta_{\alpha\beta}$. Denote the momenta conjugate to $\dot{X}$ and X' via $\Pi^\tau_\alpha \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\alpha}$, and $\Pi^\sigma_\alpha \equiv \frac{\partial \mathcal{L}}{\partial X'^\alpha}$. The variation of the above, i.e. the introduction of arbitrary $\delta X^\alpha(\tau,\sigma)$ gives the full set:

$$\begin{aligned}\delta S_{NG} &= \int_{\sigma_0}^{\sigma_f}\int_{\tau_0}^{\tau_f} d\sigma\, d\tau \left[\left(\frac{\partial}{\partial \tau}\left(\delta X^\mu\, \Pi^\tau_\mu\right)\right) + \left(\frac{\partial}{\partial \sigma}\left(\delta X^\mu\, \Pi^\sigma_\mu\right)\right) - \left(\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma}\right)\delta X^\mu\right] \\ &= \int_{\sigma_0}^{\sigma_f} \left(\delta X^\mu\, \Pi^\tau_\mu\right)\Big|_{\tau=\tau_0}^{\tau_f} + \int_{\tau_0}^{\tau_f}\left(\delta X^\mu\, \Pi^\tau_\mu\right)\Big|_{\sigma=\sigma_0}^{\sigma_f} + \int_{\sigma_0}^{\sigma_f}\int_{\tau_0}^{\tau_f} d\sigma\, d\tau\, \delta X^\mu \left(\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma}\right),\end{aligned} \tag{37.2}$$

and physically, we assume that the string's initial and final configuration is given, so that $\delta X^\mu(\tau_f,\sigma) = \delta X^\mu(\tau_0,\sigma) = 0$. This kills the first boundary term in the usual way. The second boundary term, the one at the string boundaries σ_0 and σ_f, we treat more carefully. There are two basic options – first, we can leave the spatial endpoints fixed at that boundary (remember that δX^μ is a four-vector of variation, containing both temporal and spatial generalized coordiantes), corresponding to setting $\delta X(\tau, \sigma_{0,f}) = 0$, or, equivalently, requiring that:

$$\frac{\partial X^j(\tau,\sigma_0)}{\partial \tau} = \frac{\partial X^j(\tau,\sigma_f)}{\partial \tau} = 0, \tag{37.3}$$

the so-called Dirichlet boundary condition. Or we can set $\Pi^\sigma_\mu(\tau, \sigma_{0,f}) = 0$ at the boundary, a "free endpoint" boundary condition, since no constraint is then placed on the variation.

37.2 Temporal Parametrization

We still have the problem of parametrization – not clearly a problem, per se, but a choice that we can (should, morally) use to simplify the equations of motion. In terms of interpretation, setting $\tau = t$, the coordinate time, is useful, just as it was for the point particle. Our two natural four-vectors, $\frac{\partial X^\mu}{\partial \sigma}$ and $\frac{\partial X^\mu}{\partial \tau}$ become, in this gauge:

$$\frac{\partial X^\mu(t,\sigma)}{\partial \sigma} \doteq \begin{pmatrix} 0 \\ \frac{\partial \mathbf{X}}{\partial \sigma} \end{pmatrix} \qquad \frac{\partial X^\mu(t,\sigma)}{\partial t} \doteq \begin{pmatrix} c \\ \frac{\partial \mathbf{X}}{\partial t} \end{pmatrix}, \tag{37.4}$$

where we have chosen Cartesian Minkowski space-time: $dx^\mu \doteq (c\,dt, dx, dy, dz)$. Choosing one of the coordinates as the parameter is similar to our area minimization from last time. In terms of the physics, we have, at any time t, an instantaneously at rest string parametrized by σ.

In this gauge, the Nambu-Goto action becomes:

$$S'_{NG} = -\frac{T_0}{c} \int \sqrt{(\mathbf{X}' \cdot \dot{\mathbf{X}})^2 - (-c^2 + \dot{\mathbf{X}} \cdot \dot{\mathbf{X}})(\mathbf{X}' \cdot \mathbf{X}')}\, dt\, d\sigma. \tag{37.5}$$

From here, we can verify the sign under the square root – for a piece of string that is not moving, so that $\dot{\mathbf{X}} = 0$, we have a positive area functional under the square root – if we have a string at rest (for all points along the string), we expect the area of the world-line to be $c\,(t_f - t_0)(\sigma_f - \sigma_0)$.

37.3 A σ Parametrization

Given a time t, we have a one-dimensional curve, $\mathbf{X}(t, \sigma)$, and we know how to form the unit tangent vector to the curve: choose arc-length parametrization. If we let $s = \sigma$, then arc-length parametrization immediately gives $|\mathbf{X}'| = 1$. We can use this to define the transverse component of the string velocity (the utility of which will become apparent in two lines):

$$\mathbf{v}_\perp = \dot{\mathbf{X}} - (\dot{\mathbf{X}} \cdot \mathbf{X}')\, \mathbf{X}' \tag{37.6}$$

where now, finally, $\dot{\mathbf{X}} = \frac{\partial \mathbf{X}}{\partial t}$ (since we set $\tau = t$) and $\mathbf{X}' = \frac{\partial \mathbf{X}}{\partial s}$, since we choose $s = \sigma$. This is just the full spatial velocity with the component tangent to the string (the longitudinal component) projected out. The *magnitude* of the transverse velocity, then, takes the form

$$v_\perp^2 = \dot{\mathbf{X}} \cdot \dot{\mathbf{X}} - (\dot{\mathbf{X}} \cdot \mathbf{X}')^2, \tag{37.7}$$

suggesting that we can write the action in terms of $v_\perp^2$ itself. In this gauge, then, we have (keeping the constraint $\mathbf{X}' \cdot \mathbf{X}' = 1$ in mind)

$$S'_{NG} = -\frac{T_0}{c} \int dt\, ds \sqrt{-v_\perp^2 + c^2} = -T_0 \int dt\, ds \sqrt{1 - \frac{v_\perp^2}{c^2}} \tag{37.8}$$

37.3.1 Boundary Conditions in Arc-Length Parametrization

Consider the boundary condition in this parametrization – the free endpoint boundary condition, together with our pre-gauge fixed expression for $\frac{\partial \mathcal{L}}{\partial X^\sigma}$ will give four total constraints, and these tell us physically relevant facts about the ends of an open string.

Recall:

$$\begin{aligned} \Pi^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial X'^\mu} &= -\frac{T_0}{c} \frac{(\dot{X} \cdot X')\, \dot{X}_\mu - (\dot{X})^2\, X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2\, (X')^2}} \\ &= -\frac{T_0}{c} \frac{(\mathbf{X}' \cdot \dot{\mathbf{X}})\, \dot{X}_\mu - (\dot{\mathbf{X}} \cdot \dot{\mathbf{X}} - c^2)\, X'_\mu}{\sqrt{(\mathbf{X}' \cdot \dot{\mathbf{X}})^2 - (-c^2 + \dot{\mathbf{X}} \cdot \dot{\mathbf{X}})\, (\mathbf{X}' \cdot \mathbf{X}')}} \end{aligned} \tag{37.9}$$

and this becomes, in our current gauge

$$\Pi^\sigma_\mu = -\frac{T_0}{c^2} \frac{(\mathbf{X}' \cdot \dot{\mathbf{X}})\, \dot{X}_\mu - (\dot{\mathbf{X}} \cdot \dot{\mathbf{X}} - c^2)\, X'_\mu}{\sqrt{1 - \frac{v_\perp^2}{c^2}}}. \tag{37.10}$$

The vanishing of this "momentum" provides a set of one "scalar", and one "vector" (in the spatial sense) constraint. Take $\mu = 0$, for which $\dot{X}_0 = -c$ and $X'_0 = 0$, then we learn that $\mathbf{X}' \cdot \dot{\mathbf{X}} = 0$, which here tells us that *at the endpoints*, the string motion is perpendicular to the string itself – i.e. the motion of the endpoints is necessarily transverse.

The vector portion, for $\mu = 1, 2, 3$ gives (this time only the second term in the numerator contributes, the first term being zero by the above)

$$\frac{T_0}{c^2}\,\frac{c^2 - v^2}{\sqrt{c^2 - v_\perp^2}}\,\mathbf{X}' = \frac{T_0}{c^2}\,\sqrt{c^2 - v^2}\mathbf{X}', \tag{37.11}$$

where, again from above, $v_\perp^2 = v^2$ at the endpoint. Now this must be zero, and $\mathbf{X}' \neq 0$, since it is a unit vector, hence we must have $v = c$, so *the string endpoint moves at the speed of light*.

37.4 Equations of Motion

One has to be careful with gauge conditions, as we have stressed over and over (and over). They cannot necessarily be applied before variation, and our current example is no different. They can *always* be applied to equations of motion (i.e. after variation), and there is a natural parametrization of the string equations of motion (from whence it gets its name – after all, we have yet to see a string-like equation) that makes their structure familiar and their form easily solvable.

The equation of motion for an arbitarily parametrized string read:

$$\boxed{\frac{\partial}{\partial\sigma}\,\Pi^\sigma_\mu + \frac{\partial}{\partial\tau}\,\Pi^\tau_\mu = 0.} \tag{37.12}$$

Again, for arbitrary τ and σ, we can write the "momenta":

$$\begin{aligned}\Pi^\tau_\mu &= -\frac{T_0}{c}\,\frac{(\dot{X}\cdot X')\,X'_\mu - (X')^2\,\dot{X}_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2\,(X')^2}} \\ \Pi^\sigma_\mu &= -\frac{T_0}{c}\,\frac{(\dot{X}\cdot X')\,\dot{X}_\mu - (\dot{X})^2\,X'_\mu}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2\,(X')^2}}.\end{aligned} \tag{37.13}$$

Now suppose we take $\tau = t$, imposing static gauge. Assume, further, that it is possible to develop a σ parameter that forces the spatial vectors $\mathbf{X}'\cdot\dot{\mathbf{X}} = 0$[1]. Then the four-vector dot products read:

$$\dot{X}\cdot\dot{X} = (-c^2 + \dot{\mathbf{X}}\cdot\dot{\mathbf{X}}) \quad X'\cdot X' = \mathbf{X}'\cdot\mathbf{X}' \quad \dot{X}\cdot X' = \dot{\mathbf{X}}\cdot\mathbf{X}' = 0. \tag{37.14}$$

[1]This is always possible – it has nothing to do with any intrinsic σ, but rather an identification we make on a two-dimensional grid.

Our equations of motion simplify through their momenta – with these choices, we have

$$\begin{aligned}\Pi^\tau_\mu &= \frac{T_0}{c}\,\frac{(\mathbf{X}'\cdot\mathbf{X}')\,\dot{X}_\mu}{\sqrt{(c^2-v^2)\,\mathbf{X}'\cdot\mathbf{X}'}}\\ \Pi^\sigma_\mu &= -\frac{T_0}{c}\,\frac{(c^2-v^2)\,X'_\mu}{\sqrt{(c^2-v^2)\,\mathbf{X}'\cdot\mathbf{X}'}},\end{aligned} \tag{37.15}$$

using $v^2 = \dot{\mathbf{X}}\cdot\dot{\mathbf{X}}$

As usual, this is a set of four equations, for the $\mu = 0$ case, we know that $X'_0 = 0$, so the equation of motion here is

$$\frac{\partial}{\partial\tau}\left(\frac{T_0(\mathbf{X}'\cdot\mathbf{X}')}{\sqrt{(c^2-v^2)\,\mathbf{X}'\cdot\mathbf{X}'}}\right) = 0, \tag{37.16}$$

telling us that the expression in parentheses is τ-constant, i.e. equal to some arbitrary $f(\sigma)$. We can use this to *define* the σ-parametrization. One choice, for example, is arc-length, then $\mathbf{X}'\cdot\mathbf{X}' = 1$. Here, we have the full class:

$$\sqrt{\mathbf{X}'\cdot\mathbf{X}'} = \frac{f(\sigma)\,\sqrt{c^2-v^2}}{T_0}. \tag{37.17}$$

We can input this into the spatial part of (37.12),

$$\frac{\partial}{\partial\tau}\left(\frac{1}{c}\,f(\sigma)\,\dot{\mathbf{X}}\right) - \frac{\partial}{\partial\sigma}\left(\frac{T_0^2}{c}\,\frac{\mathbf{X}'}{f(\sigma)}\right) = 0. \tag{37.18}$$

Remember, $f(\sigma)$ is up to us – we can take *any* function of σ only – here's a good choice: $f(\sigma) = f_0$, a constant. Then we have a familiar reduction, the wave equation:

$$\boxed{-\frac{f_0^2}{T_0^2}\,\ddot{\mathbf{X}} + \mathbf{X}'' = 0.} \tag{37.19}$$

To complete the traditional picture, let's set the constant $f_0 = \frac{T_0}{c}$, then we recover the wave equation with propagation speed c.

We have a few side-constraints, now, so let's tabulate our results:

$$\boxed{-\frac{1}{c^2}\,\ddot{\mathbf{X}} + \mathbf{X}'' = 0 \quad \dot{\mathbf{X}}\cdot\mathbf{X}' = 0 \quad \mathbf{X}'\cdot\mathbf{X}' = 1 - \frac{1}{c^2}\,\dot{\mathbf{X}}\cdot\dot{\mathbf{X}}.} \tag{37.20}$$

Take free endpoint boundary conditions (the string is not fixed), so that at two points $\mathbf{X}'(\sigma = \sigma_0) = \mathbf{X}'(\sigma = \sigma_f) = 0$. There are now sufficient boundary and initial conditions to actually solve the string equation.

37.5 A Rotating String

Consider a string that rotates with some constant angular velocity – we can make left and right-traveling ansatze in the usual way:

$$\begin{aligned}\mathbf{X}(t,\sigma) &= \alpha\,(A\,\cos(\kappa\,(\sigma - c\,t)) + B\,\cos(\kappa\,(\sigma + c\,t)))\,\hat{\mathbf{x}} \\ &\quad + \alpha\,(F\,\sin(\kappa\,(\sigma - c\,t)) + G\,\sin(\kappa\,(\sigma + c\,t)))\,\hat{\mathbf{y}}.\end{aligned} \tag{37.21}$$

We have automatically solved the wave equation with our choice of left and right-traveling waves. Now we have the boundary condition, $\mathbf{X}'(t, \sigma = 0, \sigma_f) = 0$. Taking the $\sigma = 0$ case, we learn that $B = A$ and $G = -F$. Then imposing the boundary condition at σ_f puts a constraint on κ:

$$\kappa\,\sigma_f = n\,\pi \longrightarrow \kappa = \frac{\pi}{\sigma_f} \tag{37.22}$$

where we have chosen to set $n = 1$ (so that as $\sigma = 0 \ldots \sigma_f$, we go across the string once, our choice). Our current solution, then, is:

$$\mathbf{X}(t,\sigma) = 2\,\alpha\,\cos(\pi\,\sigma/\sigma_f)\,(A\,\cos(\pi\,c\,t/\sigma_f)\,\hat{\mathbf{x}} - F\sin(\pi\,c\,t/\sigma_f)\,\hat{\mathbf{y}})\,. \tag{37.23}$$

Now setting the condition $\mathbf{X}' \cdot \dot{\mathbf{X}} = 0$ gives the relation $A = \pm F$, and taking the negative sign to adjust the phase, we have $F = -A$ above. Finally, we use the last of the relations in (37.20) to fix the magnitude of A:

$$\mathbf{X}' \cdot \mathbf{X}' + \frac{1}{c^2}\,\dot{\mathbf{X}} \cdot \dot{\mathbf{X}} = 1 \longrightarrow A = \pm\frac{\sigma_f}{2\,\alpha\,\pi}, \tag{37.24}$$

and our final solution reads:

$$\boxed{\mathbf{X}(t,\sigma) = \frac{\sigma_f}{\pi}\,\cos\left(\frac{\pi\,\sigma}{\sigma_f}\right)\left(\cos\left(\frac{\pi\,c\,t}{\sigma_f}\right)\hat{\mathbf{x}} + \sin\left(\frac{\pi\,c\,t}{\sigma_f}\right)\hat{\mathbf{y}}\right).} \tag{37.25}$$

From this solution, we can obtain the perpendicular component of velocity – since the motion is transverse to the string already, the velocity is perpendicular to the string everywhere, and has value:

$$v_\perp = \sqrt{\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}} = c\,\cos\left(\frac{\pi\,\sigma}{\sigma_f}\right). \tag{37.26}$$

As expected, then, the ends of the string travel at the speed of light.

37.6 Arc Length Parametrization for the Rotating String

We have an essentially undetermined σ parametrization in (37.25), and it would be nice to parametrize along the string from $-L/2 \rightarrow L/2$. The trade-off: While we could solve the familiar wave equation for our σ parametrization, essentially by construction, we must return to the full set (37.12) in a arc-length parametrization – but it will be easy to read off, for example, the energy density in this form.

Arc-length parametrization is defined by $p(\sigma)$ such that $\frac{\partial \mathbf{X}}{\partial p}$ has unit magnitude. Given that we know the magnitude in σ parametrization from (37.17) with $f(\sigma)$ constant, it is easy to perform the change-of-variables:

$$\mathbf{X}' \cdot \mathbf{X}' = \frac{\partial \mathbf{X}}{\partial p} \cdot \frac{\partial \mathbf{X}}{\partial p} \left(\frac{\partial p}{\partial \sigma}\right)^2 = 1 - \frac{1}{c^2}\, \dot{\mathbf{X}} \cdot \dot{\mathbf{X}} \longrightarrow \frac{\partial p}{\partial \sigma} = \sin\left(\frac{\pi\, \sigma}{\sigma_f}\right) \quad (37.27)$$

where we have simplified the requirement $\frac{\partial p}{\partial \sigma} = \sqrt{1 - \frac{\dot{\mathbf{X}} \cdot \dot{\mathbf{X}}}{c^2}}$ by using our solution. The ODE above is easily solved – we require $p(\sigma = 0) = -\frac{L}{2}$ so that

$$p(\sigma) = -\frac{L}{2} + \frac{\sigma_f}{\pi} - \frac{\sigma_f}{\pi} \cos\left(\frac{\pi\, \sigma}{\sigma_f}\right). \quad (37.28)$$

We can solve for σ_f in terms of L by the association

$$p(\sigma_f) = \frac{L}{2} = -\frac{L}{2} + \frac{\sigma_f}{\pi} + \frac{\sigma_f}{\pi}. \quad (37.29)$$

In this p-parametrization, the string solution is

$$\mathbf{X} = -p \left(\cos\left(\frac{2\, c\, t}{L}\right) \hat{\mathbf{x}} + \sin\left(\frac{2\, c\, t}{L}\right) \hat{\mathbf{y}}\right). \quad (37.30)$$

From this form, it is clear that the magnitude of the velocity and spatial derivatives are

$$\dot{\mathbf{X}} \cdot \dot{\mathbf{X}} = \frac{4\, c^2\, p^2}{L^2} \qquad \mathbf{X}' \cdot \mathbf{X}' = 1 \quad (37.31)$$

(primes now refer to p derivatives). Once again, the end points travel at the speed of light – in this case, we have a constant (unit) magnitude p derivative along the curve, so the velocity of the endpoints *must* be c – in the previous gauge, we required $\frac{\partial \mathbf{X}}{\partial \sigma} = 0$ at the endpoints, obscuring this result.

Finally, we can return to the canonical momenta Π^τ_μ and Π^σ_μ – it is easy to verify that (37.12) is satisfied, and our immediate concern is the energy density of this configuration. By analogy with the point particle, the energy of points along the string should be related to the t-canonical momentum's zero component. We can calculate the full tensor:

$$\Pi^\tau_\mu \doteq \begin{pmatrix} -\frac{T_0}{c\sqrt{1-\frac{4p^2}{L^2}}} \\ \frac{T_0\,\dot{\mathbf{X}}}{c^2\sqrt{1-\frac{4p^2}{L^2}}} \end{pmatrix}. \tag{37.32}$$

For particles, the analagous tensor would be p_μ, and we know that $p_0 = -\frac{E}{c}$ for E the energy density – then in this case, we should associate

$$\mathcal{E} = \frac{T_0}{\sqrt{1-\frac{4p^2}{L^2}}} \tag{37.33}$$

with the energy per unit length of the string (shown in Figure 37.1). Notice that it is a function of p – the endpoints carry infinite energy, apparently, but the total energy, given by

$$E = \int_{-L/2}^{L/2} \mathcal{E}\,dp = \frac{\pi\,L\,T_0}{2}, \tag{37.34}$$

depends only on the "tension" and length, and is finite.

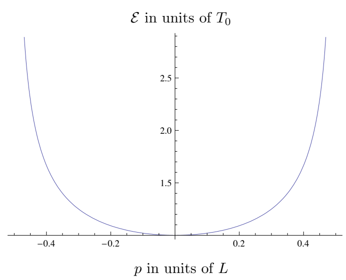

Figure 37.1: The energy density of the rotating string (37.33).