Final Examination

Classical Mechanics II
Physics 411

Date: December 9th, 2007

NAME:

WINTER BREAK MAILING ADDRESS: (if you want your exam sent)

INSTRUCTIONS: Partial credit will be given for *any* work shown on any problem. You have twenty four hours from start to finish, and you may use any book or electronic resource you find appropriate. Sign up for the oral component outside my office. Knock 'em dead.

Problem 1

(5+5)

a. Newtonian gravity is manifestly independent of test body mass – this can be used to make predictions about the effect of gravity on massless particles, although they are somewhat tenuous (a la the Newtonian "Schwarzschild radius"). Do there exist circular orbits for light in Newtonian gravity with a central, spherically symmetric body of mass M? If so, at what radii can these orbits occur?

b. General relativity provides an unambiguous language and interpretation for light-like trajectories. Do there exist circular orbits for light in Schwarzschild geometry? If so find the allowed radii, if not, prove it, and indicate physically why you cannot have such an orbit.

Problem 2

(5+5+5)

In this problem, we will develop the metric appropriate to the exterior of charged, spherical massive bodies. As with all metrics, this follows directly from Einstein's equation: $G_{\mu\nu} = 8\pi T_{\mu\nu}$.

a. The stress tensor on the right-hand side must be associated with the electric field. Since we are assuming spherical symmetry, write down a spherical ansatz for both A^μ (the electromagnetic four-potential) and $g_{\mu\nu}$, the metric.

b. From your A^μ, find the field-strength tensor $F^{\mu\nu}$ and form the stress-tensor for E&M:

$$T_{\mu\nu} = \frac{1}{4\pi}\left(g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}\right) \tag{1}$$

c. Using your spherically symmetric metric ansatz, and the spherically symmetric charged body stress tensor from above, solve Einstein's equation for both the metric *and* the electric field. Take the electric field directly from $F^{\mu\nu}$ – by going to the $r \to \infty$ limit, find the appropriate interpretation for your one constant of integration.

Problem 3

(10)

We saw that the Kerr metric could be written as:

$$
\begin{aligned}
g_{\mu\nu} &\doteq \begin{pmatrix} -\left(1-\frac{2MGr}{c^2\rho^2}\right) & 0 & 0 & -\frac{2aMGr\sin^2\theta}{c^3\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2aMGr\sin^2\theta}{c^3\rho^2} & 0 & 0 & \left((r^2+\left(\frac{a}{c}\right)^2)+\frac{2a^2MGr\sin^2\theta}{c^4\rho^2}\right)\sin^2\theta \end{pmatrix} \\
\rho^2 &\equiv r^2+\left(\frac{a}{c}\right)^2\cos^2\theta \\
\Delta &\equiv \left(\frac{a}{c}\right)^2-2\frac{MGr}{c^2}+r^2.
\end{aligned}
$$

It is clear that the above reduces to Schwarzschild in the $a=0$ case (massive, but not spinning) – characterize the space-time of the other extreme, $M=0$ but $a\neq 0$ (spinning, but not massive), set $G=c=1$ for simplicity.

Problem 4

(5+5+5)

Our ansatz for spherical symmetry led to Schwarzschild as a solution to the vacuum Einstein equations. The following line element:

$$ds^2 = -e^{2\,a(s,z)}\,dt^2 + e^{-2\,a(s,z)}\,\left(s^2\,d\phi^2 + e^{2\,b(s,z)}\,\left(ds^2 + dz^2\right)\right) \qquad (2)$$

represents a similar starting point for *cylindrically* symmetric space-times.

a. Assume $a(s,z)$ is small, and by expanding the 00 component of the metric, find the weak-field Newtonian effective potential of this metric.

b. Solve Einstein's equation outside the source – in this case, there are an infinite family of solutions parametrized by $a(s,z)$. Write your solution as a PDE for $a(s,z)$ (using the familiar flat space Laplacian in cylindrical coordinates), and the two first-order PDEs for $b(s,z)$ in terms of a and its derivatives. Does your PDE for $a(s,z)$ support your conclusion from part a.?

c. As an example, take the Newtonian potential associated with an infinite line of uniform mass density (just a flat space calculation here), find $b(s, z)$ for this choice of $a(s, z)$, and show that the resulting metric satisfies Einstein's equation away from the source.