Problem Set 11

Classical Mechanics II
Physics 411

Due on November 30th, 2007

Problem 11.1

The Schwarzschild singularity is an "apparent" singularity, $r = 2\,M$ poses no physical problems, even though the metric appears to be undefined at this point (similar to the north pole of a sphere). One way to see this is to construct invariants of the space – what is the simplest, non-zero scalar you can form that involves r, and shows that there is no intrinsic difficulty at the "Schwarzschild radius" (discounting the dimension of the space-time).

Problem 11.2

For Newtonian gravity, we have $\nabla^2\,\psi = -4\,\pi\,\psi$ as the fundamental equation governing the generation of fields from sources. Then the equations of motion follow from the Lagrangian $L = \frac{1}{2}\,m\,v^2 + m\,\psi$. A general class of solution to Laplace's equation (governing fields away from sources) is:

$$\psi = \frac{Q_0}{r} + \frac{Q_i\,r_i}{r^3} + \frac{Q_{ij}\,r_i\,r_j}{r^5} + \frac{Q_{ijk}\,r_i\,r_j\,r_k}{r^7} + \ldots. \tag{1}$$

Here, we are in flat three-dimensional space with $r_i \doteq x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$. For the first three terms, find the restriction on $Q_{\ldots}$ that gives a solution with $\nabla^2\psi = 0$. Using this, generate the most general ψ from the first three terms that has no ϕ dependence (rewrite $\mathbf{r}$ in spherical coordinates). What is this potential in the equatorial plane, $\theta = \frac{\pi}{2}$?

Problem 11.3

Find the electric potential V and magnetic vector potential $\mathbf{A}$ outside a uniformly charged rotating solid sphere (with charge density ρ, $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ constant and radius R) – see Example 5.11 in Griffiths.

a. Write your solution in terms of $\boldsymbol{\ell} = \frac{I\,\boldsymbol{\omega}}{M}$, the angular momentum per unit mass of the sphere (here I is the moment of inertia for a solid sphere) and Q, the total charge on the sphere. Put the solution in Gaussian units (most appropriate for us).

b. Now think of this as a spinning ball of mass M with uniform density ρ. Make the obvious replacement to your "electrostatic" potentials, and by the association developed in class, write down (in "spherical" coordinates) the metric perturbation associated with this configuration.

Problem 11.4

The perturbative approach we used for the ODE in precession of perihelion can also be used for polynomials. This makes some of the degeneracy issues that one encounters in such settings more explicit.

a. We have $x^2 + \epsilon\,x - 1 = 0$ for small $\epsilon \ll 1$. By taking $x = x_0 + \epsilon\,x_1$, inputting into the polynomial, and solving the equations you get at the ϵ^0 and ϵ^1 level, find the first order approximation to the roots. Check your answer using the quadratic formula and expanding the exact roots for small ϵ.

b. Try the same procedure for $\epsilon\,x^2 + b\,x + c = 0$ for $b > 0$ (find the approximate solution through order ϵ) – this time, your perturbative expansion will only find *one* root. Why? Can you modify your starting ansatz from $x = x_0 + \epsilon\,x_1$ to capture the second root? Check your answer by taking the expansion of the exact roots from the quadratic.

Problem 11.5

Find the basis vectors (orthonormal form, $\hat{\mathbf{e}}_t$, $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$) for a laboratory falling radially in towards a central body via parallel transport along the radial geodesic with boundary condition:

$$\hat{\mathbf{e}}_t \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \hat{\mathbf{e}}_r \doteq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \hat{\mathbf{e}}_\theta \doteq \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} \quad \hat{\mathbf{e}}_\phi \doteq \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r\,\sin\theta} \end{pmatrix}$$

at spatial infinity ($r \to \infty$ where the laboratory begins at rest). You are constructing the local basis vectors at all points in Schwarzschild coordinates assuming we begin with the "intuitive" laboratory frame at infinity where the Schwarzschild metric reduces to Minkowski. Observe that the $\hat{\mathbf{e}}_t$ basis vector is indeed $\dot{x}^\alpha$ for radial infall.

What are the "lengths" of these four vectors along the trajectory?

Problem 11.6

a. Show that

$$\Gamma^{\mu}{}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\,\partial_\alpha\left(\sqrt{-g}\right) \tag{2}$$

b. Recall that the stress-tensor *density* for a free-particle can be written as:

$$\mathcal{T}^{\mu\nu} = m\int \dot{x}^\mu\,\dot{x}^\nu\,\delta^4(x^\alpha - x^\alpha(s))\,ds \tag{3}$$

where s is the proper time of the particle. This is nothing but the usual $\rho\,\dot{x}^\mu\,\dot{x}^\nu$ for a degenerate mass distribution (the δ^4 ensures that there is contribution only along the world line of the particle). Form the stress tensor by appropriate introduction of $\sqrt{-g}$. As a stress tensor, we expect $T^{\mu\nu}{}_{;\mu} = 0$, show that this directly implies the geodesic equation (use the result from part a. to simplify) – it is easiest to establish the geodesic equation in the form:

$$\ddot{x}^\rho + \Gamma^{\rho}{}_{\alpha\beta}\,\dot{x}^\alpha\,\dot{x}^\beta = 0. \tag{4}$$

Problem 11.7

Plot the "cross" polarization pattern for two pairs of test masses – set the masses on the 45° lines between the x and y axes, and take an initial separation of 2 for both x and y distances. For the polarization, use $P_{xy} = 0.5$ – this is "large" by the standards of gravitational radiation, but we want to be able to see the pattern. In addition, set $\omega = 1$.