Curves and Surfaces

Lecture 14

Physics 411
Classical Mechanics II

September 28th 2007

The Riemann tensor that we saw last time is an important tool for characterizing spaces, but we can say more about the physical interpretation of it by considering its contractions. Specifically, today we will look at the definition of curvature – formally this is the "only" natural scalar we can form from the full Riemann tensor, so it is of some interest as an object, but more explicitly, the curvature scalar (or "Ricci scalar") does measure the "curvature" of a space.

In the space-time of general relativity, we lose the obvious pictorial representation we can attach to "curves" because the space that is curved isn't purely spatial, and in addition, the interpretation we will discuss today relies on an embedding in Euclidean space. We don't generally think of a curved spacetime as embedded in some large flat space (although it can be done) so the ideas discussed today are only meant to get us thinking, not to be taken in a literal sense later on.

14.1 Curves

Consider a curve in three dimensions parametrized by λ. We trace out the curve as $\lambda : 0 \to \lambda_f$. The curve is arbitrary, but we put the usual smoothness restrictions on it.

Our first goal is to determine the length of the curve – this is a simple geometric quantity given by summing up the line element $d\tau^2$:

$$d\tau = \sqrt{\left(\frac{dx^\alpha}{d\lambda}\,d\lambda\right) g_{\alpha\beta}\left(\frac{dx^\beta}{d\lambda}\,d\lambda\right)} \to \tau(\lambda_f) = \int_0^{\lambda_f} \sqrt{\frac{dx^\alpha}{d\lambda}\,\frac{dx_\alpha}{d\lambda}}\,d\lambda. \quad (14.1)$$

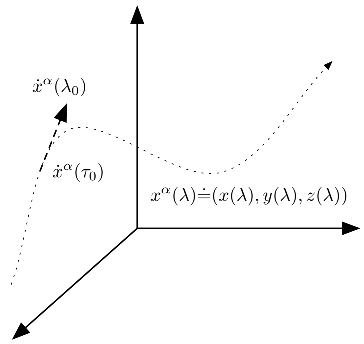

Figure 14.1: A curve, parametrized by λ.

We can parametrize the curve by τ rather than λ[1] – from the chain rule, we have

$$\frac{dx^\alpha}{d\tau} = \frac{dx^\alpha}{d\lambda}\frac{d\lambda}{d\tau} = \frac{\dot{x}^\alpha}{\sqrt{\dot{x}^\alpha\,\dot{x}_\alpha}}. \tag{14.2}$$

The λ-derivative of a curve at λ_0 is tangent to the curve at λ_0 by definition of the derivative, but simply changing the parametrization doesn't change this property – so the τ-derivative of the curve is tangent to the curve at the point τ_0. But the τ derivative has the nice property that:

$$\frac{dx^\alpha}{d\tau}\frac{dx_\alpha}{d\tau} = \frac{\dot{x}^\alpha\,\dot{x}_\alpha}{\left(\sqrt{\dot{x}^\alpha\,\dot{x}_\alpha}\right)^2} = 1 \tag{14.3}$$

so that $\frac{dx^\alpha}{d\tau} = t^\alpha$ is a *unit* tangent vector. The derivative of $t^\alpha(\tau)$ w.r.t. τ, call it k^α is perpendicular to t^α itself:

$$k^\alpha \equiv \frac{dt^\alpha}{d\tau} \to k^\alpha\,t_\alpha = \frac{dt^\alpha}{d\tau}\,t_\alpha = \frac{1}{2}\frac{d}{d\tau}(t^\alpha\,t_\alpha) = \frac{1}{2}\frac{d}{d\tau}(1) = 0 \tag{14.4}$$

so $k^\alpha(\tau_0)$ is everywhere perpendicular to the curve $x^\alpha(\tau)$.

What is this vector in terms of the derivatives of x^α w.r.t. λ, the original parametrization?

$$\boxed{\begin{aligned} k^\alpha &= \frac{dt^\alpha}{d\tau} = \frac{d\lambda}{d\tau}\frac{dt^\alpha}{d\lambda} = \frac{1}{\sqrt{\dot{x}^\alpha\,\dot{x}_\alpha}}\frac{d}{d\lambda}\left(\frac{\dot{x}^\alpha}{\sqrt{\dot{x}^\alpha\,\dot{x}_\alpha}}\right) \\ &= \frac{\ddot{x}^\alpha}{\dot{x}^\beta\,\dot{x}_\beta} - \frac{\dot{x}^\alpha\,(\ddot{x}^\gamma\,\dot{x}_\gamma)}{(\dot{x}^\beta\,\dot{x}_\beta)^2}. \end{aligned}} \tag{14.5}$$

[1] This should look more than familiar – precisely the move we make in special relativity from an arbitrary parametrization to coordinate time.

This normal vector, even though it is w.r.t. τ does not have unit magnitude, and indeed its magnitude is special, it is called the "curvature" of the curve. Let's see how this definition works with our usual ideas about curviness with a simple example.

14.1.1 Example

Take the curve to be a circle of radius r parametrized by $\lambda = \theta$:

$$x^\alpha(\theta) = \begin{pmatrix} r\,\cos\theta \\ r\,\sin\theta \\ 0 \end{pmatrix} \rightarrow \dot{x}^\alpha(\theta) = \begin{pmatrix} -r\,\sin\theta \\ r\,\cos\theta \\ 0 \end{pmatrix}. \tag{14.6}$$

We don't know what the arc-length parametrization of this curve is just by looking (or do we?!) but we can easily generate the unit tangent vector $t^\alpha = \frac{\dot{x}^\alpha}{r}$, and the normal vector, from (14.5) can be calculated:

$$\begin{aligned} \ddot{x}^\alpha(\theta) &= \begin{pmatrix} -r\,\cos\theta \\ -r\,\sin\theta \\ 0 \end{pmatrix} \\ \ddot{x}^\gamma\,\dot{x}_\gamma &= 0 \end{aligned} \tag{14.7}$$

so that we have only the first term:

$$k^\alpha = \frac{\ddot{x}^\alpha}{\dot{x}^\beta\,\dot{x}_\beta} = \begin{pmatrix} -\frac{\cos\theta}{r} \\ -\frac{\sin\theta}{r} \\ 0 \end{pmatrix} \tag{14.8}$$

and this has $\kappa^2 \equiv k^\alpha\,k_\alpha = \frac{1}{r^2}$ so the "curvature" is just $\kappa = r^{-1}$. That makes sense, if we blow the circle up, so that it has large radius, the curve looks locally pretty flat. The inverse of the curvature is called the "radius of curvature", $\kappa^{-1} = r$ and indicates the distance to the center of a circle. Locally, the radius of curvature for a more generic curve can be thought of as the radius to the center of the circle tangent to the arc segment as shown in Figure 14.2.

Finally, it is pretty easy to make the connection between the arc length and the parametrization here. If we take the curve parametrized by θ, then the

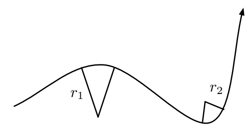

Figure 14.2: A curve with the radius of curvature for two “points” (imagine shrinking down the sector length) shown.

distance we have travelled along the curve, for a particular value of θ is $s = r\,\theta$, so we can parametrize in terms of s directly:

$$x^\alpha(s) = \begin{pmatrix} r\,\cos(s/r) \\ r\,\sin(s/r) \\ 0 \end{pmatrix} \to \dot{x}^\alpha(s) = \begin{pmatrix} -\sin(s/r) \\ \cos(s/r) \\ 0 \end{pmatrix}, \tag{14.9}$$

and we see that arc-length parametrization has automatically given us a unit tangent vector. In addition, we have $dx^\alpha\,dx_\alpha = 1$, which is to be expected – this is the spatial version of our proper time requirement: $dx^\alpha\,\eta_{\alpha\beta}\,dx^\beta = -c^2\,d\tau^2$ from special relativity.

14.2 Higher Dimension

That’s fine for one dimensional curves, but in more than one dimension, our notion of λ and τ gets confusing – there could be more than one parameter describing a surface. In addition, our ability to draw things like tangent vectors can get complicated or impossible.

The goal now is to show that the Ricci scalar plays the same role as κ in generic spaces. It certainly has the right basic form – κ was constructed out of second derivatives of a curve w.r.t. τ (what we might think of as a one-dimensional “coordinate”), and the Ricci scalar, generated out of the Riemann tensor, will involve second derivatives of the metric defining the space of interest. The similarities are nice, but the Ricci scalar defines an “intrinsic” curvature rather than the “extrinsic” curvature associated with κ. The difference is that κ requires a higher dimensional (flat) space to set the curvature – we needed two dimensions to draw Figure 14.2, while the intrinsic curvature makes no reference to an external space. The intrinsic curvature of a curve (which is necessarily one-dimensional) is zero, meaning that the curve is flat. This makes sense if you think about being trapped on the curve, with no knowledge of the exterior space – you can go forward (or back, amounting to a sign), there is only one direction, so the curve is equivalent to a line.

The Ricci scalar curvature is clearly the one of most interest to us in our study of metric spaces, since we will not have any clear embedding – we get the metric, which tells us the dimension, we do not get a higher-dimensional space from which to view. Keep this in mind as we go, while we will make contact with the κ curvature, it is a fundamentally different object.

So, we begin by thinking about the simplest two-dimensional surface, a sphere. In three dimensions, spherical coordinates still represent flat space, we are not trapped on the surface of the sphere, so our distance measurements proceed according to the usual Pythagorean rule, albeit written in funny coordinates. For a true two-dimensional surface, we cannot measure radially, and our notion of distance relies on the two-dimensional underlying metric.

We can put ourselves on the surface of the sphere by eliminating dr from the line element written in spherical coordinates:

$$ds^2 = r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2 \rightarrow g_{\mu\nu} \doteq \begin{pmatrix} r^2 & 0 \\ 0 & r^2\,\sin^2\theta \end{pmatrix} \tag{14.10}$$

This metric has non-zero connection coefficients given by:

$$\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= -\cos\theta\,\sin\theta \\ \Gamma^{\phi}_{\theta\phi} &= \Gamma^{\phi}_{\phi\theta} = \cot\theta \end{aligned} \tag{14.11}$$

and then consider the form for the Riemann tensor

$$R^{\alpha}{}_{\sigma\gamma\rho} = \Gamma^{\alpha}_{\beta\gamma}\,\Gamma^{\beta}_{\sigma\rho} - \Gamma^{\alpha}{}_{\beta\rho}\,\Gamma^{\beta}_{\sigma\gamma} - \Gamma^{\alpha}_{\sigma\gamma,\rho} + \Gamma^{\alpha}_{\sigma\rho,\gamma} \tag{14.12}$$

take $\alpha = \theta$:

$$R^{\theta}{}_{\sigma\gamma\rho} = \Gamma^{\theta}{}_{\beta\gamma}\,\Gamma^{\beta}{}_{\sigma\rho} - \Gamma^{\theta}{}_{\beta\rho}\,\Gamma^{\beta}{}_{\sigma\gamma} - \Gamma^{\theta}_{\sigma\gamma,\rho} + \Gamma^{\theta}_{\sigma\rho,\gamma} \tag{14.13}$$

and looking at the first term, only $\beta = \gamma = \phi$ contributes, so we can expand the sums over β in the first two terms

$$R^{\theta}{}_{\sigma\gamma\rho} = \Gamma^{\theta}{}_{\phi\gamma}\,\Gamma^{\phi}{}_{\sigma\rho} - \Gamma^{\theta}_{\phi\rho}\,\Gamma^{\phi}_{\sigma\gamma} - \Gamma^{\theta}_{\sigma\gamma,\rho} + \Gamma^{\theta}_{\sigma\rho,\gamma}. \tag{14.14}$$

By the symmetries of the Riemann tensor, we cannot have $\rho = \gamma$, so let's set $\rho = \theta$, $\gamma = \phi$, the only choice (keeping in mind that $\rho = \phi$, $\gamma = \theta$ is just the negative of this) is:

$$R^{\theta}{}_{\sigma\phi\theta} = \Gamma^{\theta}_{\phi\phi}\,\Gamma^{\phi}_{\sigma\theta} - \Gamma^{\theta}_{\sigma\phi,\theta} \tag{14.15}$$

from which we get potentially two terms

$$\begin{aligned} R^{\theta}_{\theta\phi\theta} &= \Gamma^{\theta}_{\theta\phi}\,\Gamma^{\phi}_{\theta\theta} - \Gamma^{\theta}_{\theta\phi,\theta} = 0 \\ R^{\theta}_{\phi\phi\theta} &= \Gamma^{\theta}_{\phi\phi}\,\Gamma^{\phi}_{\phi\theta} - \Gamma^{\theta}_{\phi\phi,\theta} = (-\cos\theta\,\sin\theta)\,(\cot\theta) + \sin^2\theta - \cos^2\theta = -\sin^2\theta \\ &= -R^{\theta}_{\phi\theta\phi} \end{aligned} \tag{14.16}$$

For $\alpha = \phi$ in (14.12), we have

$$R^{\phi}{}_{\sigma\gamma\rho} = \Gamma^{\phi}_{\beta\gamma}\,\Gamma^{\beta}_{\sigma\rho} - \Gamma^{\phi}{}_{\beta\rho}\,\Gamma^{\beta}{}_{\sigma\gamma} - \Gamma^{\phi}_{\sigma\gamma,\rho} + \Gamma^{\phi}_{\sigma\rho,\gamma} \tag{14.17}$$

and as before, our only option is $\rho = \theta$, $\gamma = \phi$:

$$R^{\phi}{}_{\sigma\phi\theta} = \Gamma^{\phi}_{\beta\phi}\,\Gamma^{\beta}{}_{\sigma\theta} - \Gamma^{\phi}{}_{\beta\theta}\,\Gamma^{\beta}{}_{\sigma\phi} - \Gamma^{\phi}{}_{\sigma\phi,\theta} + \Gamma^{\theta}{}_{\sigma\theta,\phi} \tag{14.18}$$

the first term is zero for both $\beta = (\theta,\phi)$, and the fourth term is zero since there is no ϕ dependence in the metric. We have (the second term above only contributes for $\beta = \phi$):

$$R^{\phi}{}_{\sigma\phi\theta} = -\Gamma^{\phi}{}_{\phi\theta}\,\Gamma^{\phi}{}_{\sigma\phi} - \Gamma^{\phi}{}_{\sigma\phi,\theta} \tag{14.19}$$

which can only be non-zero for $\sigma = \theta$, then the only non-zero component left is

$$R^{\phi}{}_{\theta\phi\theta} = -\Gamma^{\phi}_{\phi\theta}\,\Gamma^{\phi}_{\theta\phi} - \Gamma^{\phi}_{\theta\phi,\theta} = -\cot^2\theta + \frac{1}{\sin^2\theta} = 1 \tag{14.20}$$

Collecting all of this, we have two non-zero components of the Riemann tensor:

$$\begin{aligned} R^{\theta}{}_{\phi\phi\theta} &= -R^{\theta}{}_{\phi\theta\phi} = -\sin^2\theta \\ R^{\phi}_{\theta\phi\theta} &= -R^{\phi}_{\theta\theta\phi} = 1 \end{aligned} \tag{14.21}$$

The Ricci tensor is formed from this via:

$$R_{\sigma\rho} = R^{\alpha}{}_{\sigma\alpha\rho} \tag{14.22}$$

so that in matrix form, we have

$$\begin{aligned} R_{\sigma\rho} &\doteq \begin{pmatrix} R^{\theta}{}_{\theta\theta\theta} + R^{\phi}{}_{\theta\phi\theta} & R^{\theta}{}_{\theta\theta\phi} + R^{\phi}{}_{\theta\phi\phi} \\ R^{\theta}{}_{\phi\theta\theta} + R^{\phi}{}_{\phi\phi\theta} & R^{\theta}{}_{\phi\theta\phi} + R^{\phi}{}_{\phi\phi\phi} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \end{aligned} \tag{14.23}$$

and then finally, the Ricci scalar is:

$$\begin{aligned} R &= g^{\mu\nu}\, R_{\mu\nu} = g^{\theta\theta}\, R_{\theta\theta} + g^{\phi\phi}\, R_{\phi\phi} \\ &= \frac{1}{r^2} + \frac{1}{r^2\, \sin^2\theta}\, \sin^2\theta \\ &= \frac{2}{r^2}. \end{aligned} \tag{14.24}$$

That was a pretty explicit calculation, and we went through it carefully this time so you could see how such arguments go – with all the indices and entries, the use of symmetries and a quick scan of the relevant indices that will play a role in the final answer is important.

Notice that $R = \frac{2}{r^2}$ is a quantity we could measure from the surface of the sphere itself – we could measure the Riemann tensor by taking a vector and going around different paths, and then it would be a simple matter to discover the curvature of our two-dimensional space. Alternatively, we could make distance measurements in different directions in order to construct the metric, and then use that to find the curvature. Either way, we make no reference to any higher dimensional space in which we are embedded.

14.3 Taking Stock

Let's collect and summarize the elements we have been defining and working on. In particular, we shall review the covariant derivative, its role in defining geodesics, and also in determining the curvature of space.

Our one and two-dimensional curves and surfaces have been useful in understanding concepts like curvature, but now we kick the scaffold over and are on our own. We will not be referring to the space(-times) in GR as embedded in some higher dimensional space, so things like Ricci curvature are what we get, with no reference to the second derivatives w.r.t. parametrizations of curves.

14.3.1 Properties of the Riemann Tensor

We started by defining the Riemann tensor in terms of a lack of commutativity in second (covariant) derivatives. Remember:

$$\begin{aligned} f^\alpha{}_{;\beta} &= f^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\, f^\gamma \\ f_{\alpha;\beta} &= f_{\alpha,\beta} - \Gamma^\gamma{}_{\alpha\beta}\, f_\gamma, \end{aligned} \tag{14.25}$$

and these covariant derivatives are defined by: 1. Our desire to have

$$f^{\alpha'}{}_{;\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x^\sigma}{\partial x'^\beta} f^\gamma{}_{;\sigma} \tag{14.26}$$

and 2. The requirement that $(f^\alpha\, f_\alpha)_{;\gamma} = (f^\alpha\, f_\alpha)_{,\gamma}$.

These two ideas lead to derivatives that re-define our notion of displacement (that's what vectors *do* after all), and in particular, we generated a new type of "constant" vector – one that is parallel-transported along a curve (with tangent $\dot{x}^\gamma$) via

$$\frac{D}{D\tau} f^\alpha = f^\alpha{}_{;\beta}\, \dot{x}^\beta = 0 \tag{14.27}$$

a set of ordinary differential equations that tells us what the value of $f^\alpha(x(\tau))$ is given $f^\alpha(x(0))$ – in flat space, the corresponding vector is just a constant, but here in curved space, we must consider how the vector transforms from point-to-point.

There is a special class of curve – geodesics, defined in terms of parallel transport as "curves whose tangent vector is parallel-transported around themselves". Formally, this amounts to replacing f^α with $\dot{x}^\alpha$ above

$$\frac{D}{D\tau}\dot{x}^\alpha = \dot{x}^\alpha{}_{;\beta}\,\dot{x}^\beta = \dot{x}^\beta\left(\dot{x}^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\,\dot{x}^\gamma\right) = \ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\,\dot{x}^\beta\,\dot{x}^\gamma = 0 \tag{14.28}$$

and geometrically, the same equation comes from a variational principle $\delta \int ds = 0$ indicating that solutions are "straight lines".

After that, we developed the requirement that the angle between vectors transported in the above manner should be constant along the curve, and used this to uniquely determine $\Gamma^\alpha{}_{\beta\gamma}$ in terms of our metric $g_{\mu\nu}$

$$g_{\mu\nu;\gamma} = 0 \to \Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\rho}\left(g_{\rho\beta,\gamma} + g_{\rho\gamma,\beta} - g_{\beta\gamma,\rho}\right). \tag{14.29}$$

With a definite relationship between the (derivatives of the) metric and the connection, we were able to ask the question: How is $f^\alpha{}_{;\beta\gamma}$ related to $f^\alpha{}_{;\gamma\beta}$

and answer it in terms of more derivatives of the metric – this led us to the definition of the Riemann tensor

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv \Gamma^{\alpha}{}_{\sigma\gamma}\,\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\sigma\delta}\,\Gamma^{\sigma}{}_{\beta\gamma} + \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta}. \tag{14.30}$$

and we saw how this is connected to the difference of two vectors transported around two different curves.

But looking at the above – it is clear that, for example, there is a symmetry w.r.t. $\gamma \leftrightarrow \delta$ – the situation becomes more interesting when we lower the α:

$$R_{\alpha\beta\gamma\delta} \equiv \Gamma_{\alpha\sigma\gamma}\,\Gamma^{\sigma}{}_{\beta\delta} - \Gamma_{\alpha\sigma\delta}\,\Gamma^{\sigma}{}_{\beta\gamma} + g_{\alpha\tau}\left(\Gamma^{\tau}{}_{\beta\delta,\gamma} - \Gamma^{\tau}{}_{\beta\gamma,\delta}\right), \tag{14.31}$$

and we ask: What are the symmetries here? Clearly, $\gamma \leftrightarrow \delta$ is still antisymmetric, but what more can we say?

14.3.2 Normal Coordinates

I want to make the usual observation about coordinate systems and tensors. This sort of "reasoning" guides a lot of derivations and is, in a way, the method for making a non-tensorial theory into a tensor theory (which one needs to do a lot in GR). The statement is: if you can make a tensor statement in any coordinate system, it holds in all coordinate systems.

This is not an unfamiliar procedure – think of the electric field of a dipole $\mathbf{p} = p_0 \hat{\mathbf{z}}$ sitting at the origin:

$$\mathbf{E} = \frac{p_0}{4\,\pi\,\epsilon_0\,r^3}(2\,\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{14.32}$$

From this expression, we can form the coordinate-free form:

$$\mathbf{E} = \frac{1}{4\,\pi\,\epsilon_0\,r^3}(3\,(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}), \tag{14.33}$$

and we have turned a result in a specific coordinate system into a coordinate-independent statement.

The same is true for a lot of expressions that come up in general relativity: We find some easy set of coordinates, prove whatever it is we want in terms of those, and then make the non-tensor statement (usually) into a tensor statement at which point it's true in any coordinates. Let me construct a particularly nice coordinate system and use it to study the Riemann tensor.

Consider a point P in space (or space-time). Suppose we know the geodesics of the space, and we travel along one of them from P at parameter $\tau = 0$ to a nearby point P' at τ. The geodesic satisfies the defining equation (14.28), and the Taylor expansion near P is given by

$$x^\alpha(\tau) = x^\alpha(0) + \tau\,\dot{x}^\alpha(0) - \frac{1}{2}\,\tau^2\,\Gamma^\alpha_{\ \beta\gamma}(P)\,\dot{x}^\beta(0)\,\dot{x}^\gamma(0) + O(\tau^3). \tag{14.34}$$

Suppose we want to change coordinates, in particular, we want to consider coordinates $x'^\alpha(0) = \tau\,\dot{x}^\alpha(0)$. Well, then all geodesics emanating from P can be written in the transformed coordinates:

$$\frac{d\dot{x}'^\alpha}{d\tau} + \Gamma'^\alpha_{\ \beta\gamma}(P)\,\dot{x}'^\beta\,\dot{x}'^\gamma = 0 \tag{14.35}$$

for any geodesic, now. Using the transformation, $\frac{dx'^\alpha}{d\tau} = \dot{x}^\alpha(0)$, a constant, so the first term above is zero, and the Christoffel term is:

$$\Gamma'^\alpha_{\ \beta\gamma}(P)\,\dot{x}'^\beta(0)\,\dot{x}'^\gamma(0) = 0. \tag{14.36}$$

We could have used any geodesic we wanted to generate this equation and it applies to any geodesic passing through P, so the only way it can be zero is if $\Gamma'^\alpha_{\ \beta\gamma}(P) = 0$ (if you don't believe me, do the explicit transformation of the Christoffel symbols). As for the metric,

$$g'_{\mu\nu;\gamma}(P) = g'_{\mu\nu,\gamma} - \Gamma'^\sigma_{\ \mu\gamma}\,g'_{\sigma\nu} - \Gamma'^\sigma_{\ \nu\gamma}\,g'_{\mu\sigma} = g'_{\mu\nu,\gamma} = 0, \tag{14.37}$$

so that $g'_{\mu\nu}$ is constant at the point P.

Actually calculating the normal coordinates for a given system is not necessarily easy to do. It requires, for example, that you have some nice form for the geodesic trajectories. But just knowing that such a coordinate system exists can get us pretty far[2]. For example, consider the Riemann tensor, defined in (14.30). In these normal coordinates, the two terms quadratic in Γ are zero, but the derivatives are not, so we have:

$$\begin{aligned}
R^\alpha_{\ \rho\gamma\beta}|_{NC} &= \Gamma^\alpha_{\ \beta\rho,\gamma} - \Gamma^\alpha_{\ \gamma\rho,\beta} \\
R_{\alpha\rho\gamma\beta}|_{NC} &= \Gamma_{\alpha\beta\rho,\gamma} - \Gamma_{\alpha\gamma\rho,\beta} \\
&= \frac{1}{2}\left(g_{\alpha\beta,\rho\gamma} - g_{\beta\rho,\alpha\gamma} - g_{\alpha\gamma,\rho\beta} + g_{\gamma\rho,\alpha\beta}\right).
\end{aligned} \tag{14.38}$$

[2]The prescription is: set the connection to zero at a point, set the metric to the identity matrix (with ± 1 along the diagonal) to put the equation in normal coordinates. More than that, one cannot do, the second derivatives of the metric are not in general zero (unless the space is flat).

If we can make tensor statements, then our choice of normal coordinates is a moot point – that's the real idea here. Well, looking at the above, I see that in these coordinates, the Riemann tensor is antisymmetric under: $\alpha \leftrightarrow \rho$ and $\gamma \leftrightarrow \beta$ and symmetric under $(\alpha, \rho) \leftrightarrow (\gamma, \beta)$. The tensor statement is, for example:

$$R_{\alpha\rho\gamma\beta}|_{NC} + R_{\rho\alpha\gamma\beta}|_{NC} = 0, \tag{14.39}$$

and since this is a tensor equation (addition of two Riemann tensors), and its zero in one coordinates system, it must be zero in all – we can drop the NC identifier.

The symmetries just mentioned are true for the generic expression (14.30), but here we can see it clearly. There are more – consider cyclic permutations, again referring to (14.38), we see that

$$\boxed{R_{\alpha\rho\gamma\beta} + R_{\alpha\gamma\beta\rho} + R_{\alpha\beta\rho\gamma} = 0} \tag{14.40}$$

(this is not as obvious as the rest, but can be worked out relatively quickly). Again, since it is a tensor statement, this holds in all coordinate systems.

This exhausts the symmetries of the Riemann tensor, and we're finally ready to do the famous counting argument. As a four-indexed object in D dimensions, we have *a priori* D^4 independent components. But consider an antisymmetric tensor $A_{\mu\nu}$, this has $\frac{1}{2}D(D-1)$ components. Now take a symmetric tensor $S_{\mu\nu}$ in N dimensions, there are $\frac{1}{2}N(N+1)$ components there – viewing $R_{(\alpha\beta)(\gamma\delta)}$ as R_{AB}, a symmetric tensor (since $R_{(\alpha\beta)(\gamma\delta)} = R_{(\gamma\delta)(\alpha\beta)}$ and each pair $A = (\alpha\beta)$, $B = (\gamma\delta)$ is antisymmetric – I use parenthesis here only to group) in $N = \frac{1}{2}D(D-1)$ dimensions, we have:

$$R_{\alpha\beta\gamma\delta} \text{ components } = \frac{1}{2}\left(\frac{1}{2}D\,(D-1)\right)\left(\frac{1}{2}D\,(D-1)+1\right) - \text{Number in (14.40)} \;. \tag{14.41}$$

To count the number of constraints imposed by (14.40), notice that if one sets any two components equal, we get zero identically by the symmetries already in place (for example, take $\rho = \alpha$: one term goes away by antisymmetry, the other two cancel), so only for $(\alpha\rho\gamma\beta)$ distinct do we get a constraint. Well, there are "D choose 4" ways to arrange the indices,

$$\begin{aligned}
\text{components} &= \frac{1}{2}\left(\frac{1}{2}\,D\,(D-1)\right)\left(\frac{1}{2}\,D\,(D-1)+1\right)-\left(\begin{array}{c} D \\ 4 \end{array}\right) \\
&= \frac{1}{8}\,D\,(D-1)\,(D\,(D-1)+2)-\frac{1}{24}\,D\,(D-1)\,(D-2)\,(D-3) \\
&= \frac{1}{12}\,D^2\,(D^2-1)
\end{aligned} \tag{14.42}$$

so that in our three-dimensional space, there are only 6 independent components in the Riemann tensor, whereas in four-dimensional space-time, there are 20.

Finally, I mention the derivative relationship for the Riemann tensor, this is also easiest to show in the normal coordinates we have been considering. If we take a derivative in (14.38), then

$$R_{\alpha\rho\gamma\beta,\delta} = \frac{1}{2}\left(g_{\alpha\beta,\rho\gamma\delta} - g_{\beta\rho,\alpha\gamma\delta} - g_{\alpha\gamma,\rho\beta\delta} + g_{\gamma\rho,\alpha\beta\delta}\right) \tag{14.43}$$

and we can cyclically permute the last three indices:

$$R_{\alpha\rho\gamma\beta,\delta} + R_{\alpha\rho\delta\gamma,\beta} + R_{\alpha\rho\beta\delta,\gamma} = 0. \tag{14.44}$$

Now this is not a tensor statement, but in normal coordinates, a normal derivative is equal to a covariant one (since the connection vanishes), so replacing the commas in the above with semicolons, we have the "Bianchi Identity" – this is important in deriving Einstein's equation.

14.4 Summary

In setting up these elements of tensor analysis, we have at certain points specialized to a class of spaces. The arena of general relativity is a space with zero torsion (Christoffel symbols are symmetric) and a "metric connection" (connection related to derivatives of the metric) which automatically implies that the metric has zero covariant derivative. One thing that can be shown for these spaces is that at a point P, the Christoffel connection can itself be made zero (think of a connection coefficient at a point, and a coordinate transformation at P – this can be constructed so that the connection vanishes), so at any point P, we can set up coordinates such that

$$g_{\alpha\beta;\gamma} = 0 \text{ with } \Gamma^{\alpha}{}_{\beta\gamma} = 0 \rightarrow g_{\alpha\beta,\gamma} = 0, \tag{14.45}$$

that is: at the point P, the metric itself is constant – so we can construct at any point a flat metric with zero connection (more than that we cannot do, the second derivatives of the metric will not vanish in general). That's important for doing physics, since we believe, for the most part, that our local environment is flat space. This belief comes from wide experience and *must* be built in, in some manner, to any physical theory. For the spaces we are discussing, we see that this is natural.