Equivalence Principles

Lecture 15

Physics 411
Classical Mechanics II

October 1st 2007

We have the machinery of tensor analysis, at least enough to discuss the physics – we are in a four-dimensional space-time with a metric. From the metric we can calculate connections and curvature, but how to relate these to the physical picture of masses orbiting central bodies? That is: how does mass *generate* curvature? Perhaps a step back from this is the question, *why* does mass generate curvature? Or even: *does* mass generate curvature?

This final question is the most reasonable starting point – my goal here is to make this question precise and begin to answer it. Historically, the important observation (for the physics) is the weak equivalence principle.

15.1 Newtonian Gravity

Newtonian gravity consists of two parts: a theory of motion under a central potential:

$$\boxed{m\,\ddot{x}^{\alpha} = -m\,\phi_{,}{}^{\alpha}} \tag{15.1}$$

and a theory of the potential that relates it to a source of mass:

$$\boxed{\nabla^2\,\phi = 4\,\pi\,\rho} \tag{15.2}$$

(we set $G = 1$, in units where $c = 1$) very much equivalent to the potential and charge density relation in electrostatics.

The first equation is a statement about the effect of the field on a particle of mass m, and the second is a statement about the generation of the field by a source ρ. Our studies in general relativity have, thus far, focused on the theory of motion of a particle given the field (the metric $g_{\mu\nu}$).

This has been somewhat downplayed – our discussions involved general Riemannian spaces – but these are precisely the ones of physical interest: they are completely defined by a metric, with a metric connection and (hence) Riemann tensor related to derivatives of the metric. The metric is the *only* input, and it is the field of interest. Now we have specialized further to four dimensions with one "temporal" coordinate, but that doesn't change much.

Before we begin with equivalence principles, it is important to reiterate the underlying problem with the each of the above components – the motion portion leads to different predictions of force in different inertial frames, and the relation of ρ to ϕ through the Laplacian violates finite propagation speed (time is not involved at all). While one can write a (special) relativistic theory for a potential, there is yet a third problem with the "field" equation of Newtonian gravity: it can only couple to explicitly massive sources. That is to say, we have no way of translating energy into ρ here. We must do this, since mass and energy are equivalent, every form of energy has got to couple to gravity.

Each of these will be addressed as we develop the complete general relativistic form of gravitation – we expect to have to modify the equations of motion for a test particle, and the field equations themselves – in addition, the field equations must allow coupling to all forms of energy.

15.2 Equivalence

15.2.1 Weak Equivalence Principle

Early on, in our study of physics, when we talk about the acceleration of gravity near the earth, we write:

$$F = m_i\, a = -m_p\, g \tag{15.3}$$

(with m_i the inertial mass, m_p the passive mass, "charge") – and find that all objects accelerate due to gravity according to the value of g, not the values of m_i and m_p. It is easy to forget how surprising this result is, though. After all, for generic potential ϕ_g, we have the force F_g:

$$F_g = -m_p\, \nabla\, \phi_g. \tag{15.4}$$

Compare with the electric force, $F_E = -q\, \nabla\, \phi$ – apparently m_p is playing the role of q, the charge. That has, in principle, nothing to do with the mass of the body m_i, just as q is unrelated to the mass of the body. But of course, to great experimental accuracy, m_i and m_p *are the same*. This somewhat surprising observation forms the basis of the weak equivalence principle, $m_i = m_p$. It tells us that, for example, it is impossible to distinguish between a local gravitational field (like the one nearby the earth, with $a = g$) and an accelerated object – the famous elevators of GR: an accelerated elevator in empty space has physics (dropping balls) which looks like an elevator at rest on the earth's surface. As observers of the falling ball, the earth is the anomaly here – it holds us fixed. If we were falling with the balls, we would observe nothing! So the question is – is the ball falling due to a gravitational field or "just" due to a uniform acceleration that we do not experience? You can't tell. Again, because the mass doesn't change the acceleration, we fall at the same rate as the balls (if the pesky earth didn't get in the way), so they would appear to be at rest.

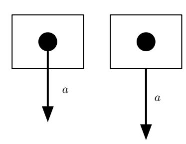

Figure 15.1: Two views: the ball falls due to the force of gravity or the ball falls because it is in a box undergoing constant acceleration.

So no observer can tell, through ball dropping anyway, whether the local area surrounding the experiment has gravity, or just uniform acceleration. Obviously, gravity as a phenomenon is not always constant, so if the elevator were big enough, you could see the effects of perturbations in the gravitational field (the uniform gravitational acceleration we feel at the surface of the earth is just an approximation). But this sort of consideration leads us to the next section. The point here is that: a. everyone feels the same thing and b. that thing can be approximated locally as uniform acceleration. Of course it gets worse (better) – even energy (light, say) feels gravity.

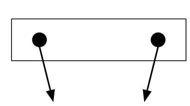

Figure 15.2: For an extremely large elevator, and over extreme distances, two balls in a box approach each other – evidently, to distinguish between constant acceleration and gravity one needs to compare *two* falling objects.

What does this mean for physics? Well, the idea was to make lemonade – since all objects experience the same acceleration, re-interpret that acceleration as a feature of the arena, the space-time we are in. We eliminate the force of gravity in favor of a curved space whose geodesics (falling objects) play the role of straight lines. The weak equivalence principle gives us the freedom to do this, since we can view the acceleration as either due to a force, or due to a uniform acceleration, pick the road less traveled.

15.2.2 Strong Equivalence Principle

That's all well and good, we take the force and stuff it into the geometry so we can ignore it. But how do we determine the geometry generated by a matter distribution? And what constraints must we place on this geometry so that it squares with the previous half century of physical understanding? These questions are answered by the "strong equivalence principle" which states that locally, it must be possible to imagine that we are in a four-dimensional flat space-time with physics governed by special relativity, that is, we must be able to define locally inertial Minkowski frames. Among other things, this ensures that special relativity holds. At that point, though, what you are saying (and what Einstein did say) is that gravity should be locally undetectable. But that seems silly, you're sitting on those chairs for a reason. What he has in mind is free-fall – the earth itself is the "problem" here, a giant floor on which to stand isn't the point. And again this is how curvature enters the picture, we fall freely along geodesics – from a flat space point of view, orbiting the sun is acceleration, but trapped on the surface of a sphere, orbiting the sun is traveling in a straight line.

The sleight-of-hand we have performed was to specialize the discussion of geometry so much that the "spaces" we have been studying have local flatness built in. That specialization was motivated by these equivalence principles. The idea that we have a metric, that at any point P in the space, we can set that metric to a flat one and kill off the Christoffel connection (at the point) is already in place. Even the deviation vector of parallel transport, defined in terms of the Riemann tensor is second order in the path lengths – to first order, space is flat. This is a feature we built-in to the discussion: Rather than focusing on general spaces, we zeroed in on the ones which obey the Strong Equivalence Principle.

15.3 The Field Equations

So what to do? I have suggested that we need to look to deviations of two balls being dropped (for example) to distinguish between a frame of constant acceleration and any sort of gravity. Basically, we will take the analysis as far as we can go, and then appeal to "minimal substitution". In the end, you can't *derive* Einstein's equation any more than you can derive Newton's laws – they can be motivated, but they must be verified by physical observations.

15.3.1 Equations of Motion

Let's turn these equivalence principles into some physics. One of the implications of the strong equivalence principle is that we should write down all our physics as tensor equations. That way, no matter where we are, we get the proper transformations out of the physical laws. Seems sensible – let's do the easiest case. A free particle in a flat space-time has acceleration:

$$\frac{dv^\alpha}{d\tau} = 0, \tag{15.5}$$

because it involves non-covariant derivatives, this is not a viable equation, but the correction (as we have belabored) is easy:

$$\frac{Dv^\alpha}{D\tau} = 0, \tag{15.6}$$

and this just says that free motion occurs along geodesics of the curved space-time – that is also true for the non-covariant form, but the geodesics of flat space are straight lines. Great circles on a sphere, again, are "straight lines" in that curved spatial setting.

What should we do about the metric field? How can we combine the equivalence principles to make a concrete statement about the metric? Another question which would be natural to ask if we hadn't narrowed our view of geometry is: Is the metric field the only thing that matters? The answer in our setting is "yes", and this again is by construction. Given the metric, we can construct its derivatives to get $\Gamma^\alpha{}_{\beta\gamma}$ and take *its* derivatives to get $R^\alpha{}_{\beta\gamma\delta}$, then work our way through the rest of the list of important tensors. We are looking, then, for a field equation to set constraints on $g_{\mu\nu}$.

To do that, we must somehow involve the metric in our discussion, and this is not so easy to do. After all, we claim that locally, gravity isn't observable, everyone travels along their geodesics and no one is the wiser. True, but only in small patches. Suppose we are on an elevator freely falling towards the center of the earth, we drop two balls along with ourselves. We would see the balls (which look, over a small range, like they are falling parallel to one another) begin to approach each other (think of a radially-directed field pointing towards each ball from some external center). Similarly, in a curved space, the way in which the balls approach each other would tell us something about the curvature of the space.

Let's be more explicit. A geodesic in flat space is a straight line – if we have two straight lines that cross at a point, and you and a friend start there and walk along your geodesics, you notice the distance between the two of you increasing (linearly). For two people trapped on the surface of a sphere, you start together at the south pole, say, and walk along your geodesics – the distance between you increases at first, but then decreases until you meet again at the north pole. In a sense, we can measure curvature in this manner, by violating the "small local patch" assumption of the equivalence principle. So we consider the classical and relativistic form of the "geodesic deviation equations".

Newtonian Deviation

On the classical side, we have a gravitational field defined by a potential ϕ and connected to a matter distribution ρ by $\nabla^2\phi = \rho$. We take two test particles in this field defined by their coordinates $x_1^\alpha(\tau)$ and $x_2^\alpha(\tau)$, then the equations of motion for the two particles are

$$\begin{aligned} \ddot{x}_1 &= -\nabla\phi \\ \ddot{x}_2 &= -\nabla\phi. \end{aligned} \tag{15.7}$$

Consider the situation at time t_0, then we are at $x_1(t_0)$ and $x_2(t_0)$ and there, the accelerations read:

$$\begin{aligned} \ddot{x}_1(t_0) &= -\nabla\,\phi(x_1(t_0)) \\ \ddot{x}_2(t_0) &= -\nabla\,\phi(x_2(t_0)) = -\nabla\,\phi(x_1(t_0) + \eta(t_0)) \end{aligned} \tag{15.8}$$

where we have defined $\eta \equiv x_2 - x_1$, the separation vector. Assuming the curves are relatively close together, we take η to be small, and expand the potential about the first point $x_1(t_0)$

$$\ddot{x}_2(t_0) = -\nabla\,\phi(x_1(t_0)) + \eta\cdot\nabla\left(\nabla\,\phi(x_1(t_0)\right) = \ddot{x}_1(t_0) - \eta\cdot\nabla\left(\nabla\,\phi(x_1(t_0))\right) \tag{15.9}$$

from which we conclude that classically,

$$\boxed{\ddot{\eta} = -\eta \cdot \nabla (\nabla \phi)\,.} \tag{15.10}$$

15.4 Geodesic Deviation in a General Space-Time

For the same problem treated relativistically, we need to do a little more setup (basically, it's just a bit harder to do the Taylor expansion). Consider two geodesics "close together", as shown in Figure 15.3 – we have two natural vectors, the tangent to the curve at constant σ (σ is a curve-selecting parameter): $\dot{x}^\alpha(\tau, \sigma) \equiv \frac{\partial x^\alpha(\tau,\sigma)}{\partial \tau}$, and the orthogonal σ derivative $x'^\alpha(\tau, \sigma) \equiv \frac{\partial x^\alpha(\tau,\sigma)}{\partial \sigma}$. We should be careful – these two directions form a two-dimensional surface on our space (read space-time), that they are orthogonal is something we need to establish. Well, suppose they were orthogonal at some point τ_0, then by our metric connection $g_{\alpha\beta;\gamma} = 0$, we know that they will be orthogonal at all points along (either) curve. I claim (without proof) that these two vector fields can be made orthogonal at a point and hence are orthogonal everywhere.

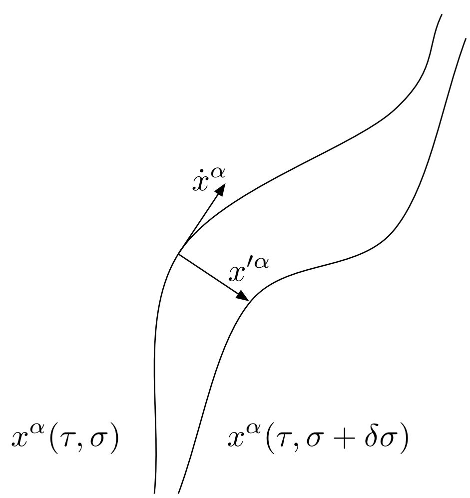

Figure 15.3: Two geodesic curves: $x^\alpha(\tau, \sigma)$ and $x^\alpha(\tau, \sigma + \delta\sigma)$

Now, with this coordinate system in place (two-dimensional, a τ direction along the geodesic and a σ direction moving us to some other geodesic), we can calculate the displacement vector η^α at a constant τ:

$$\eta^\alpha = \frac{\partial x^\alpha}{\partial \sigma}\, d\sigma. \tag{15.11}$$

We are interested in how this vector changes as we move along the curve $x^\alpha(\tau, \sigma)$, that's a question answered by our covariant derivative:

$$\begin{aligned} \frac{D\eta^\alpha}{D\tau} &= \frac{D}{D\tau}\left(\frac{\partial x^\alpha}{\partial \sigma}\right) d\sigma = \left(\frac{\partial^2 x^\alpha}{\partial\sigma\partial\tau} + \Gamma^\alpha{}_{\beta\gamma}\,\frac{\partial x^\beta}{\partial \sigma}\,\frac{\partial x^\gamma}{\partial\tau}\right) d\sigma \\ &= \frac{D}{D\sigma}\left(\frac{\partial x^\alpha}{\partial\tau}\right) d\sigma. \end{aligned} \tag{15.12}$$

This is what we would call the "velocity" of the deviation, to get the acceleration (in order to compare with the Newtonian form) we need to take one more τ derivative

$$\frac{D}{D\tau}\left(\frac{D\eta^\alpha}{D\tau}\right) = \left(\frac{D}{D\tau}\left(\frac{D\dot{x}^\alpha}{D\sigma}\right) - \frac{D}{D\sigma}\left(\frac{D\dot{x}^\alpha}{D\tau}\right) + \frac{D}{D\sigma}\left(\frac{D\dot{x}^\alpha}{D\tau}\right)\right) d\sigma, \tag{15.13}$$

just a simple addition and subtraction, but the first two terms look a lot like a commutator of covariant derivatives. Indeed, for a vector $f^\alpha(\tau, \sigma)$ defined on our surface via parallel propagation, we have (defining $x'^\alpha \equiv \frac{\partial x^\alpha}{\partial\sigma}$, $\dot{x}^\alpha \equiv \frac{\partial x^\alpha}{\partial \tau}$)

$$\begin{aligned} \frac{D}{D\tau}\left(\frac{Df^\alpha}{D\sigma}\right) &= f^\alpha_{;\beta\gamma}\, x'^\beta\, \dot{x}^\gamma + f^\alpha{}_{;\gamma}\,\frac{Dx'^\gamma}{D\tau} \\ \frac{D}{D\sigma}\left(\frac{Df^\alpha}{D\tau}\right) &= f^\alpha_{;\beta\gamma}\, \dot{x}^\beta\, x'^\gamma + f^\alpha{}_{;\gamma}\,\frac{D\dot{x}^\gamma}{D\sigma} \end{aligned} \tag{15.14}$$

but from (15.12), the last term on the right in each of the above are equal, so when we subtract, we get

$$\begin{aligned} \frac{D}{D\tau}\left(\frac{Df^\alpha}{D\sigma}\right) - \frac{D}{D\sigma}\left(\frac{Df^\alpha}{D\tau}\right) &= \left(f^\alpha_{;\beta\gamma} - f^\alpha_{;\gamma\beta}\right)\, \dot{x}^\beta\, x'^\gamma \\ &= -R^\alpha{}_{\rho\gamma\beta}\, x'^\gamma\, \dot{x}^\beta\, f^\rho \end{aligned} \tag{15.15}$$

where the last line follows from the definition of the Riemann tensor. Now setting $f^\alpha = \dot{x}^\alpha$ we can finish off (15.13),

$$\boxed{\frac{D^2\eta^\alpha}{D\tau^2} = -R^\alpha{}_{\rho\gamma\beta}\,\dot{x}^\beta\,\dot{x}^\rho\,\eta^\gamma,} \tag{15.16}$$

and the last term in (15.13) is zero since $\dot{x}^\alpha$ is parallel-transported along x^α (these are geodesics).

Remember the goal here, we want to compare (15.16) to (15.10), but what we've got now are two different expressions for the deviation vector η in two different spaces (three dimensional flat-space, and some unspecified four-dimensional space-time). But if we squint, then $\ddot{\eta}_\beta = -(\eta \cdot \nabla (\nabla \phi)) = -\eta^\alpha \phi_{,\alpha\beta}$, and we might convince ourselves that a reasonable association is $\phi_{,\alpha\beta} \approx R_{\alpha\rho\beta\gamma}\, \dot{x}^\gamma\, \dot{x}^\rho$.

That's all well and good, but what should we do with the "source" equation – the Newtonian potential ϕ that appears in (15.10) is sourced by a matter distribution ρ via Poisson's equation $\nabla^2\phi = 4\,\pi\,\rho$. I'll motivate this in a second, let me just say for now that ρ viewed as matter-energy density can be written in terms of a stress-energy tensor $T_{\alpha\beta}$ via $\rho \approx T_{\alpha\beta}\, v^\alpha\, v^\beta$ for an "observer" (the particle reacting to ρ) traveling with four-velocity v^α. This begs the final association:

$$\nabla^2\phi = \phi_{,\alpha}{}^{\alpha} \approx R^\alpha{}_{\rho\alpha\gamma}\, \dot{x}^\gamma\, \dot{x}^\rho \approx 4\pi\, T_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta \Rightarrow R_{\rho\gamma} \approx 4\,\pi\, T_{\rho\gamma}. \tag{15.17}$$

I leave the $\approx$ symbol in because, and I want to stress this, the above equation is incorrect. One might also consider it to be . . . unrigorous. You can't deny that it is exactly what we want, though – precisely a relationship between the metric, its derivatives, and a matter source. Indeed the jump from the above to Einstein's final equation (and apparently he wrote down the above as well) is a short one. Regardless of your outlook, in terms of how we got here, let its accuracy be our guide, we will be convinced when we see the correct physics come out.

15.4.1 Aside: Stress-Energy Tensors

We know from Newtonian gravity that a (stationary) matter distribution $\rho(x)$ (energy density, more generally) generates a Newtonian field. In E&M, a charge distribution generates the electric field and current generates the magnetic – in four-vector language we have $j^\alpha(x)$ (zero component is $\rho(x)$, spatial component is $\mathbf{j}(x)$) – in terms of what sort of source GR should have, these are both options. But going back to E&M, the fields $\mathbf{E}$ and $\mathbf{B}$ *also* form a "source", now a tensor, called the stress-energy (or energy-momentum) tensor. Apart from the matter itself, electric and magnetic fields have energy, momentum, etc. Now GR clearly must involve massive sources, but mass and energy are equivalent, so the field equations of general relativity must couple to energy as well. This means that $T_{\alpha\beta}$, the full energy-momentum tensor must be involved.

For electrodynamics, the full four dimensional stress-energy tensor gives us a set of conservation laws encapsulated in the divergence condition ${T^{\mu\nu}}_{;\mu} = 0$. These four equations, which behave like ${J^{\mu}}_{;\mu} = 0$ for charge conservation, effectively give us the relations between energy density and the Poynting vector, and the Poynting vector and Maxwell stress tensor (the spatial components of $T^{\mu\nu}$).

In special relativity, we learn how to make an observation of density – and we will see this later on, but the ρ component of the most general distribution $T^{\mu\nu}$ are observed by contracting the stress tensor with the observer's four-velocity (the local temporal basis vector in the observer's rest frame). This means that an energy density ρ must involve the four-velocities of particles measuring it.

15.5 Einstein's Equation

All right, let's go back to the geodesic deviation correspondence that we had previously:

$$R_{\mu\nu} \approx 4\,\pi\,T_{\mu\nu}, \tag{15.18}$$

if we take the covariant divergence of both sides, we should get zero (by the definition of energy-momentum tensors – this property is not unique to the Maxwell case). What can we say about the derivatives of the Ricci tensor?

Remember we had the Bianchi identity for the Riemann tensor:

$$R_{\alpha\rho\gamma\beta;\delta} + R_{\alpha\rho\delta\gamma;\beta} + R_{\alpha\rho\beta\delta;\gamma} = 0 \tag{15.19}$$

but we want the Ricci tensor form of this statement, so hit the above with $g^{\alpha\gamma}$:

$$R_{\rho\beta;\delta} - R_{\rho\delta;\beta} + {R^{\gamma}}_{\rho\beta\delta;\gamma} = 0 \tag{15.20}$$

and multiplying this by $g^{\rho\delta}$, for example, gives

$$\begin{aligned} 0 &= R^{\rho}{}_{\beta;\rho} - R^{\rho}{}_{\rho;\beta} + R^{\gamma\rho}{}_{\beta\rho;\gamma} \\ &= R^{\rho}{}_{\beta;\rho} - R_{;\beta} + R^{\gamma}{}_{\beta;\gamma} \\ &= 2\, R^{\rho}{}_{\beta;\rho} - R_{;\beta} \end{aligned} \tag{15.21}$$

so we learn that $R^{\mu}{}_{\nu;\mu} = \frac{1}{2}\, R_{;\nu}$ which we can write as

$$R^{\mu\nu}{}_{;\mu} = \frac{1}{2}\left(R g^{\mu\nu}\right)_{;\mu} \rightarrow \left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R\right)_{;\nu} = 0 \tag{15.22}$$

So the cute trick: Suppose we replace the lone Ricci tensor on the left of (15.18) with the combination above (and for correspondence reason, we also multiply the right-hand side by 2)

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 8\,\pi\, T_{\mu\nu}. \tag{15.23}$$

This is Einstein's equation – the tensor on the right is called the Einstein tensor:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R,} \tag{15.24}$$

and this tells us, given a distribution of "source" (as typified by the energy-momentum tensor), how to construct $g_{\mu\nu}$ – notice that although it is simple to write down the equation, the left-hand side involves quadratic derivatives in the metric and is highly non-linear (remember that the Riemann tensor itself has terms like $\Gamma\Gamma$, which are quadratic in first derivatives of the metric). In general, solving Einstein's equation exactly as written is almost impossible. So as we go along, we will do two important things to make life tractable. The first thing we can do is simplify the form of the metric based on physical arguments (symmetries, for example). Then there's also vacuum solutions, where we move away from the matter itself (or other fields) and consider the source-free solutions.