Introduction to Fields

Lecture 16

Physics 411
Classical Mechanics II

October 3rd, 2007

We introduce the definition of a field, which leads almost immediately to the mathematical description of the simplest scalar fields. We connect the field equations to equations of motion familiar from classical mechanics, and make the association between field Lagrangians and particle Lagrangians, the latter can be viewed as a continuum limit of the former.

Our goal, over the next few classes, is to look at how all of the machinery of Lagrange and Hamiltonian analysis carries over to scalar, vector and tensor fields. Along the way, we will encounter familiar objects from the most familiar fields: **E** and **B**.

16.1 Lagrangians for Fields

Consider a spring connecting two masses in one dimension. The location of the left-mass we'll call x_{-1} and the location of the right x_1. Our goal is to find the time-dependence of the motion of the two masses: $x_1(t)$ and $x_{-1}(t)$.

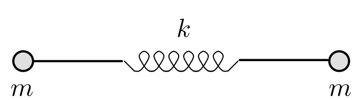

Figure 16.1: Two identical masses connected by a spring.

We can form the Lagrangian, the kinetic energy is just

$$T = \frac{1}{2}\, m\, \dot{x}_1^2 + \frac{1}{2}\, m\, \dot{x}_{-1}^2 \tag{16.1}$$

as always for a two-mass system. The potential energy depends on the extension of the spring: $x_1 - x_{-1}$ and the equilibrium length associated with the spring, call it a:

$$U = \frac{1}{2}\, k\, ((x_1 - x_{-1}) - a)^2, \tag{16.2}$$

so that

$$L = T - U = \frac{1}{2}\, m \left(\dot{x}_1^2 + \dot{x}_{-1}^2\right) - \frac{1}{2}\, k\, ((x_1 - x_{-1}) - a)^2 \tag{16.3}$$

and then the equations of motion can be obtained in the usual way:

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}_1} - \frac{\partial}{\partial x_1}\right) L = m\, \ddot{x}_1 + k\, ((x_1 - x_{-1}) - a) \\ 0 &= \left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}_{-1}} - \frac{\partial}{\partial x_{-1}}\right) L = m\, \ddot{x}_{-1} - k\, ((x_1 - x_{-1}) - a). \end{aligned} \tag{16.4}$$

The above can be solved by introducing a center-of-mass coordinate: $x_1 + x_{-1}$ and a difference coordinate $x_1 - x_{-1}$.

If we introduce more springs, the same basic procedure holds – take three masses, now we have three coordinates, labelled x_{-1}, x_0 and x_1. The La-

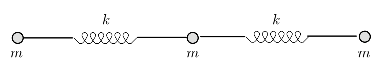

Figure 16.2: Three masses connected by two identical springs.

grangian undergoes the obvious modification:

$$L = \frac{1}{2}\, m \left(\dot{x}_{-1}^2 + \dot{x}_0^2 + \dot{x}_1^2\right) - \frac{1}{2}\, k \left(((x_1 - x_0) - a)^2 + ((x_0 - x_{-1}) - a)^2\right), \tag{16.5}$$

and the equations of motion follow. In particular, consider the equation for $x_0(t)$

$$m\, \ddot{x}_0 - k\, (x_{-1} - 2\, x_0 + x_1) = 0. \tag{16.6}$$

Notice that for this "internal" mass, there is no mention of the equilibrium position a. As we add more springs and masses, more of the equations of motion will depend only on the relative displacements on the left and right. In the end, only the boundary points (the left and right-most masses, with no compensating spring) will show any a-dependence.

16.1.1 The Continuum Limit for Equations of Motion

What we want to do, then, is switch to new coordinates that make the Lagrangian manifestly independent of a. We can accomplish this by defining a set of local relative positions. Take a uniform grid $\{\bar{x}_j\}_{j=-N}^{N}$ with $\bar{x}_j \equiv ja$ for a the grid spacing, and introduce the variables $\{\phi(x_j, t)\}_{j=-N}^{N}$ to describe the displacement from $\bar{x}_j$ as shown in Figure 16.3, so that

$$\phi(x_j, t) \equiv x_j(t) - \bar{x}_j \tag{16.7}$$

in terms of our previous coordinates $\{x_j(t)\}_{j=-N}^{N}$.

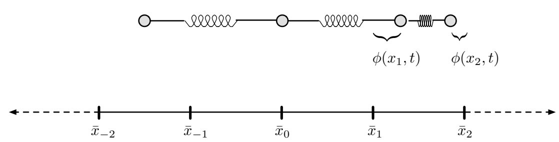

Figure 16.3: A grid with uniform spacing a and a few of the displacement variables $\phi(x_j, t)$ shown.

Transforming the Lagrangian is simple – for each of the terms in the potential, we have

$$((x_j(t) - x_{j-1}(t)) - a)^2 = (\phi(x_j, t) - \phi(x_{j-1}, t))^2 \tag{16.8}$$

and the kinetic terms are replaced via $\dot{x}_j(t) = \dot{\phi}(x_j, t)$. So the Lagrangian in these new generalized coordinates, for a grid with spacing a becomes

$$L = \frac{1}{2} m \sum_{j=-N}^{N} \dot{\phi}(x_j, t)^2 - \frac{1}{2} k \sum_{j=-N}^{N} (\phi(x_j, t) - \phi(x_{j-1}, t))^2 . \tag{16.9}$$

The equations of motion become, ignoring the boundary points (i.e. assume we are looking at a finite segment of an infinite chain):

$$m \ddot{\phi}(x_j, t) - k (\phi(x_{j+1}, t) - 2 \phi(x_j, t) + \phi(x_{j-1}, t)) = 0, \tag{16.10}$$

capturing the sentiment of (16.6). The force term in parentheses can be expanded as a series in a

$$\begin{aligned}\phi(x_{j+1},t) &- 2\phi(x_j,t) + \phi(x_{j-1},t) \\ &\approx \left(\phi(x_j,t) + a\,\frac{\partial \phi(x_j,t)}{\partial x_j} + \frac{1}{2}\,a^2\,\frac{\partial^2 \phi(x_j,t)}{\partial x_j^2}\right) \\ &- 2\phi(x_j,t) + \left(\phi(x_j,t) - a\,\frac{\partial \phi(x_j,t)}{\partial x_j} + \frac{1}{2}\,a^2\,\frac{\partial^2 \phi(x_j,t)}{\partial x_j^2}\right) \\ &= a^2\,\frac{\partial^2 \phi(x_j,t)}{\partial x_j^2} + O(a^4),\end{aligned} \tag{16.11}$$

so that we may write the equation of motion as

$$m\,\ddot{\phi}(x_j,t) - k\,a^2\,\phi''(x_j,t) = 0 \tag{16.12}$$

with $\phi'(x_j,t) \equiv \frac{\partial \phi(x_j,t)}{\partial x_j}$. This is an approximate result for a, the grid spacing, small. We are now prepared to take the final step, passing to the limit $a \to 0$, so that we are describing a continuum of springs and balls. The mathematical move is relatively simple here, but the moral point is considerable: we are promoting $\phi(x_j,t)$, a continuous function of t labelled by x_j to a continuous function of x, i.e. position. We will then have a *field*, a continuous function that assigns, to each point x, at each time t, a displacement value.

Suppose we consider a finite segment of an infinite chain with fixed total mass M and fixed total length L. We expect $M = \mu\,L$, with μ the mass density (here just mass per unit length). Looking at the equation of motion, we see that dividing through by a will give $\frac{m}{a}$ for the first term, that's precisely the μ we want.

$$\frac{m}{a}\,\ddot{\phi}(x,t) - k\,a\,\phi''(x,t) = 0 \tag{16.13}$$

The second term has the factor $k\,a$ – evidently, if we want to take $a \to 0$ to yield a reasonable limit, we must shrink down the mass of each individual ball so as to keep $\frac{m}{a} \equiv \mu$ constant, and at the same time, increase the spring constant of the springs connecting the balls so that $k\,a$ remains constant.

It is pretty reasonable to assume a finite length and finite mass as we shrink $a \to 0$, but what physical description should we give to the flexibility of the rod? After all, we are building a continuous object out of springs, so shouldn't the rod itself have some associated spring constant?[1] Think of the

[1] This physical property is "Young's modulus" in the theory of elasticity.

"net" spring constant for two springs as in Figure 16.2 – the equations of motion are:

$$\begin{aligned} m\,\ddot{x}_{-1} &= k\,(x_0 - x_{-1} - a) \\ m\,\ddot{x}_0 &= -k\,(x_0 - x_{-1} - a) + k\,(x_1 - x_0 - a) \\ m\,\ddot{x}_1 &= -k\,(x_1 - x_0 - a). \end{aligned} \tag{16.14}$$

To find the effective spring constant, we "remove" the second mass (that is, set the middle m to zero), then we can solve for x_0 and input in both the top and bottom equations. The equation for the total stretch is then defined by $x_1 - x_{-1}$, which we can obtain by subtracting the top from the bottom in the above:

$$m\,(\ddot{x}_1 - \ddot{x}_{-1}) = -\frac{2\,k^2}{k+k}\,(x_1 - x_{-1} - 2\,a) \tag{16.15}$$

which is the correct equation of motion for a spring with effective spring constant: $k_{eff} = \frac{k^2}{k+k} = \left(\frac{1}{k} + \frac{1}{k}\right)^{-1}$, and equilibrium spacing $2\,a$. For our setup with $2\,N+1$ particles, we have $2\,N$ springs, and the effective spring constant is

$$k_{eff} = \left(\sum_{j=1}^{2N} \frac{1}{k}\right)^{-1} = \frac{k}{2\,N}. \tag{16.16}$$

From this, we can understand the constant $k\,a$ from (16.13) – we send $k \to \infty$ as $N \to \infty$ (or in other words, $a \to 0$) in order to keep the effective spring constant of the continuum rod the same for all N. Let's replace the local k in (16.13) with k_{eff}, our constant:

$$\frac{m}{a}\ddot{\phi}(x,t) - 2\,N\,k_{eff}\,a\,\phi''(x,t) = 0. \tag{16.17}$$

In terms of the constant length L, we have $L = a\,(2\,N)$ and the constant mass $M = (2\,N+1)\,m$, so we can write the field equation in terms of N as

$$\frac{M}{2\,N+1}\frac{2\,N}{L}\,\ddot{\phi}(x,t) + 2\,N\,k_{eff}\,\frac{L}{2\,N}\,\phi''(x,t) = \frac{M}{L}\,\frac{2\,N+1}{2\,N}\,\ddot{\phi}(x,t) + k_{eff}\,L\,\phi''(x,t) = 0, \tag{16.18}$$

and now the limit is simple: take $N \to \infty$, define $\mu \equiv \frac{M}{L}$ and we have:

$$\ddot{\phi}(x,t) = \frac{k_{eff}\,L}{\mu}\,\phi''(x,t). \tag{16.19}$$

Notice that the constant $k_{eff}\,L$ looks like a force (tension, for example) and $k_{eff}\,L/\mu$ has units of velocity squared. What we have is a wave equation representing longitudinal propagation of displacement, a slinky if you like. If we define $v^2 \equiv \frac{k_{eff} L}{\mu}$, then the solutions to the above are the usual

$$\phi(x,t) = f_\ell(x + v\,t) + f_r(x - v\,t) \tag{16.20}$$

so that given some initial displacement function, $\phi(x,0) = f_\ell(x) + f_r(x)$, we know how the displacement propagates as a function of time. We started with balls and springs, and ended with a function $\phi(x,t)$ that tells us, for each time t, by how much a bit of slinky is displaced from its "equilibrium", the local compression and stretching of the slinky as a function of time. More concretely, $\phi(x,t)$ tells us where the mass that *should* be at x (in equilibrium) *is* relative to x.

16.1.2 The Continuum Limit for the Lagrangian

Given that we have an equation of motion for a simple scalar field, it is natural to return to the Lagrangian, and try to find a variational principle that generates the field equation. It is reasonable to ask why we are interested in a Lagrangian – after all, we have the field equation (16.19), and its most general solution, why bother with the Lagrangian from whence it came? The situation is analagous to mechanics – in general one starts with free-body diagrams and Newton's laws, and then moves to a Lagrangian description (then Hamiltonian, etc.) – the motivation is coordinate freedom. On the field side, the same situation holds – Lagrangians and variational principles provide a certain generality, in addition to a more compact description.

From a modern point of view, field theories are *generated* by their Lagrangians. Most model building, extensions of different theories, occurs at the level of the theory's action, where interpretation is somewhat simpler, and the effect of various terms are well-known in the field equations. Finally, perhaps the best motivation of all, we recover a field-theoretic version of Noether's theorem.

So, we return to (16.9),

$$L = \frac{1}{2}\,m \sum_{j=-N}^{N} \dot\phi(x_j,t)^2 - \frac{1}{2}\,k \sum_{j=-N+1}^{N} \left(\phi(x_j) - \phi(x_{j-1},t)\right)^2, \tag{16.21}$$

and consider a typical approximation to an integral – for a function $f(\bar x)$ shown in Figure 16.4, if we take the right-hand edge of the box defined by

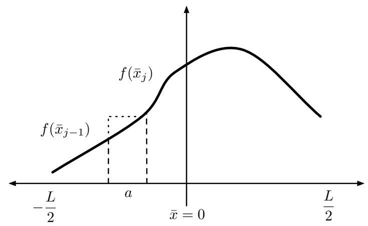

Figure 16.4: A function $f(\bar{x})$ on a grid with spacing a.

a flat bottom of width a and a height $f(\bar{x}_j)$, then the area of the box is $f(\bar{x}_j) \times a$, and if we sum up all of these boxes:

$$\sum_{j=-N}^{N} a\, f(\bar{x}_j) \approx \int_{-L/2}^{L/2} f(\bar{x})\, d\bar{x}, \tag{16.22}$$

where the approximation gets better as $a \to 0$ (alternatively $N \to \infty$).

We can write the discrete Lagrangian to take advantage of this – using Taylor expansion on the potential term, and factoring,

$$L = \frac{1}{2}\frac{m}{a} \sum_{j=-N}^{N} a\, \dot{\phi}(x,t)^2 - \frac{1}{2}\, k \sum_{j=-N}^{N} \big(a\, \phi'(x_j,t)\big)^2 \tag{16.23}$$

and introducing the constants M, L and k_{eff},

$$L = \frac{1}{2}\frac{M}{L}\, \frac{2N}{2N+1} \sum_{j=-N}^{N} a\, \dot{\phi}(x,t)^2 - \frac{1}{2}\, k_{eff}\, 2\, N\, \frac{L}{2\,N} \sum_{j=-N}^{N} a\big(\phi'(x_j,t)\big)^2 \tag{16.24}$$

we can take the limit as $N \to \infty$

$$\begin{aligned} L &= \frac{1}{2}\, \mu \int_{-L/2}^{L/2} \dot{\phi}(\bar{x},t)^2\, d\bar{x} - \frac{\mu\, v^2}{2} \int_{-L/2}^{L/2} \phi'(\bar{x},t)^2\, d\bar{x} \\ &= \int_{-L/2}^{L/2} \frac{\mu}{2} \left(\dot{\phi}(\bar{x},t)^2 - v^2\, \phi'(\bar{x},t)^2\right) d\bar{x}. \end{aligned} \tag{16.25}$$

16.2 Multi-Dimensional Action Principle

We begin from the Lagrangian

$$L = \frac{\mu}{2} \int_{-L/2}^{L/2} \left(\dot{\phi}(x,t)^2 - v^2 \phi'(x,t)^2 \right) dx \tag{16.26}$$

and will form an action $S = \int L dt$ that will allow us to define a scalar function $\mathcal{L}$, the Lagrange density as the integrand of a double integral, time and space together. This function is more naturally a "Lagrangian" in field theory (because it lacks integrals), and the variation of the action will produce precisely the field equation we saw earlier, but entirely within the continuum context.

Consider the action for the Lagrangian (16.26):

$$S[\phi(x,t)] = \int_{t_0}^{t_f} \int_{-L/2}^{L/2} \frac{\mu}{2} \left(\dot{\phi}(x,t)^2 - v^2\, \phi'(x,t)^2 \right) dx dt. \tag{16.27}$$

Time and space are both integrated, and S is, as always, a functional – it takes a function of (x,t) to the real numbers. Our usual particle action does the same thing:

$$S_\circ[x(t)] = \int_{t_0}^{t_f} \left(\frac{1}{2}\, m\, \dot{x}(t)^2 - U(x) \right) dt \tag{16.28}$$

but the integration is only over t, and $x(t)$ is a function only of t. Just as we call the integrand of $S_\circ[x(t)]$ the Lagrangian L, it is customary to call the integrand of (16.27) $\mathcal{L}$, the "Lagrange Density". This is a density in the sense that we integrate it over space, in addition to time, so that $\mathcal{L}$ has the units of Lagrangian-per-unit-length.

One Dimensional Variation

Remember the variational principle from single-particle classical mechanics – we vary a trajectory (in this case), keeping the endpoints fixed. That is, we add to $x(t)$ an arbitrary function $\eta(t)$ that vanishes at t_0 and t_f, so that $\tilde{x}(t) = x(t) + \eta(t)$ has the same endpoints as $x(t)$, as shown in Figure 16.5.

Our variational requirement is that S is, to first order, unchanged by an arbitrary $\eta(t)$. This is the usual "stationary" or "extremal" action statement

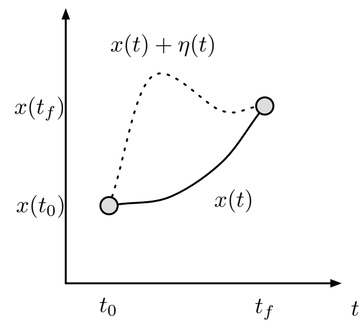

Figure 16.5: A trajectory $x(t)$ with a perturbed trajectory $x(t)+\eta(t)$ sharing the same endpoints.

– let's review the idea. If we expand

$$\begin{aligned} S_\circ[x(t)+\eta(t)] &= \int_{t_0}^{t_f}\left(\frac{1}{2}\,m\,(\dot{x}+\dot{\eta})^2 - U(x(t)+\eta(t))\right)\,dt \\ &\approx \int_{t_0}^{t_f}\left(\frac{1}{2}\,m\,(\dot{x}^2+2\,\dot{x}\,\dot{\eta}) - \left(U(x(t)) + \frac{dU}{dx}\,\eta(t)\right)\right)\,dt \end{aligned} \tag{16.29}$$

by keeping only those terms first order in $\eta(t)$ (and its derivatives), then we can collect a portion that looks like $S_\circ[x(t)]$ and terms that represent deviations:

$$S_\circ[x(t)+\eta(t)] \approx S_\circ[x(t)] + \underbrace{\int_{t_0}^{t_f}\left(m\,\dot{x}\,\dot{\eta} - \frac{dU}{dx}\,\eta\right)\,dt}_{\equiv \delta S}. \tag{16.30}$$

The extremization condition is that $\delta S = 0$ for any $\eta(t)$, but in order to see what this implies about $x(t)$, we must first write the perturbation entirely in terms of $\eta(t)$, i.e. we must replace $\dot{\eta}(t)$. An integration by parts on the first term of δS does the job:

$$\int_{t_0}^{t_f} \dot{x}\,\dot{\eta}\,dt = \underbrace{\dot{x}(t)\,\eta(t)\Big|_{t=t_0}^{t_f}}_{=0} - \int_{t_0}^{t_f} \ddot{x}\,\eta(t)\,dt \tag{16.31}$$

with the boundary term vanishing at both endpoints by assumption ($\eta(t_0) = \eta(t_f) = 0$).

So transforming the $\dot{\eta}$ term in δS,

$$\delta S = \int_{t_0}^{t_f} \left(-m\,\ddot{x} - \frac{\partial U}{\partial x} \right) \eta(t)\,dt \tag{16.32}$$

and for this to be zero for any $\eta(t)$, the integrand itself must be zero, and we recover:

$$m\,\ddot{x} = -\frac{\partial U}{\partial x}. \tag{16.33}$$

16.2.1 Two Dimensional Variation

Variation in two dimensions, x and t is no different. Referring to Figure 16.6, we have some function $\phi(x,t)$ with associated action $S[\phi(x,t)]$ and we want this action to be unchanged (to first order) under the introduction of an arbitrary additional function $\eta(x,t)$ with $\phi(x,t)$ and $\phi(x,t)+\eta(x,t)$ sharing the same boundary conditions, i.e. $\eta(x,t)$ vanishes on the boundaries.

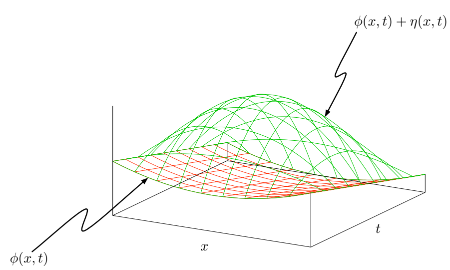

Figure 16.6: Variation in two dimensions – we have a surface $\phi(x,t)$ and the perturbed surface $\phi(x,t)+\eta(x,t)$ sharing the same boundaries in both x and t.

The action, to first order in $\eta(x,t)$ and its derivatives is:

$$\begin{aligned} S[\phi(x,t)+\eta(x,t)] &\approx \int_{t_0}^{t_f} \int_{-L/2}^{L/2} \frac{\mu}{2} \left(\dot{\phi}^2(x,t) + 2\,\dot{\phi}\dot{\eta} - v^2 \left(\phi'(x,t)^2 + 2\,\phi'(x,t)\,\eta'(x,t) \right) \right) dxdt \\ &= S[\phi(x,t)] + \int_{t_0}^{t_f} \int_{-L/2}^{L/2} \mu \left(\dot{\phi}\,\dot{\eta} - v^2\,\phi'\,\eta' \right) dxdt, \end{aligned} \tag{16.34}$$

where, as above, the second term here is δS, and we want this to vanish for arbitrary $\eta(x,t)$. Again, we use integration by parts, now on two terms, to rewrite $\dot{\eta}$ and η' in terms of η itself. Noting that

$$\int_{t_0}^{t_f} \int_{-L/2}^{L/2} \dot{\phi}\, \dot{\eta}\, dx dt = \underbrace{\int_{-L/2}^{L/2} \dot{\phi}(x,t)\, \eta(x,t)\Big|_{t=t_0}^{t_f} dx}_{=0} - \int_{t_0}^{t_f} \int_{-L/2}^{L/2} \ddot{\phi}(x,t)\, \eta(x,t)\, dx dt, \tag{16.35}$$

and, similarly, that

$$\int_{t_0}^{t_f} \int_{-L/2}^{L/2} \phi'\, \eta'\, dx dt = -\int_{t_0}^{t_f} \int_{-L/2}^{L/2} \phi''(x,t)\, \eta(x,t)\, dx dt. \tag{16.36}$$

We can write the perturbation to the action as

$$\delta S = \int_{t_0}^{t_f} \int_{-L/2}^{L/2} \mu \left(-\ddot{\phi} + v^2\, \phi'' \right) \eta(x,t)\, dx dt \tag{16.37}$$

and as before, for this to hold for all $\eta(x,t)$, the integrand must vanish, giving

$$\boxed{\ddot{\phi}(x,t) = v^2\, \phi''(x,t)} \tag{16.38}$$

the wave equation.

Where is the potential here? It is interesting that what started as particles connected by strings appears to have become a "free" scalar field in the sense that there is no "force" in the field equations, nor a term like $U(\phi)$ in the action. In other words, there is no "source" for the field here. That is true, and we will deal with that later on. For now, our next job is to build the Euler-Lagrange equations and generic action.

16.3 Higher Dimensional Actions

We have a particular example of a free scalar field, and this serves as a model for more general situations. An action like (16.27) depends on the derivatives of the field $\phi(x,t)$. Notice that it depends on the derivatives in an essentially antisymmetric way. This suggests that if we introduce the usual coordinates $x^\mu \doteq (v\,t, x)$ for $\mu = 0, 1$, and a metric:

$$g_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16.39}$$

(a metric with Lorentzian signature) then the action can be written as

$$S = -\int\int \frac{\mu\, v}{2}\, \phi_{,\mu}\, g^{\mu\nu}\, \phi_{,\nu}\, d(v\, t) dx \tag{16.40}$$

with $\phi_{,\mu} \equiv \frac{\partial \phi}{\partial x^\mu} \equiv \partial_\mu \phi$ as always. In this form, the Lagrange density is

$$\mathcal{L} = -\frac{\mu\, v}{2}\, \phi_{,\mu}\, g^{\mu\nu}\, \phi_{,\nu} \tag{16.41}$$

and one typically writes $d\tau \equiv dx^0 dx^1 = d(vt)dx$, so the action takes the form

$$\boxed{S = \int \mathcal{L}\, d\tau.} \tag{16.42}$$

The advantage to the metric notation is that it becomes clear how more spatial dimensions should be handled – we can trivially introduce, for example, the Minkowski metric in Cartesian coordinates, and two additional spatial derivatives appear automatically. We can also consider more complicated Lagrange densities. Our current example involves only the derivatives of ϕ, so we would write $\mathcal{L}(\phi_{,\mu})$, but what happens to the field equations that come from a Lagrange density of the form $\mathcal{L}(\phi, \phi_{,\mu})$?

For the moment, let's ignore the overall constants in the action, those are important, but the field equations do not depend on them. If we had, in any $D+1$ dimensional space, the action

$$S[\phi(x^\mu)] = \int \mathcal{L}(\phi, \phi_{,\mu})\, d\tau \tag{16.43}$$

with $d\tau \equiv \prod_{\nu=0}^{D} dx^\nu$ (Cartesian and temporal) and we extremize using an arbitrary $\eta(x^\mu)$ that vanishes on the boundaries, then

$$\begin{aligned} S[\phi(x^\mu) + \eta(x^\mu)] &= \int \mathcal{L}(\phi + \eta, \phi_{,\mu} + \eta_{,\mu})\, d\tau \\ &\approx \int \left(\mathcal{L}(\phi, \phi_{,\mu} + \eta_{,\mu}) + \frac{\partial \mathcal{L}(\phi, \phi_{,\mu} + \eta_{,\mu})}{\partial \phi} \eta \right) d\tau \\ &\approx \int \mathcal{L}(\phi, \phi_{,\mu})\, d\tau + \underbrace{\int \left(\frac{\partial \mathcal{L}(\phi, \phi_{,\mu})}{\partial \phi}\, \eta + \frac{\partial L(\phi, \phi_{,\mu})}{\partial \phi_{,\mu}}\, \eta_{,\mu} \right) d\tau}_{\equiv \delta S}\, . \end{aligned} \tag{16.44}$$

The δS term has a summation in it: $\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\, \eta_{,\mu}$, and we need to use integration by parts on each term of the sum. The sum contains all derivatives of $\mathcal{L}$

w.r.t. each derivative of ϕ dotted into the corresponding derivative of η – we can flip the derivatives one by one leaving us with boundary terms which vanish and an overall ∂_μ acting on the partials $\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}$. Simply put,

$$\int \frac{\partial \mathcal{L}(\phi, \phi_{,\mu})}{\partial \phi_{,\mu}} \eta_{,\mu} \, d\tau = - \int \eta \, \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \phi_{,\mu})}{\partial \phi_{,\mu}} \right) d\tau. \tag{16.45}$$

To extremize the action, we enforce $\delta S = 0$ for arbitrary $\eta(x^\mu)$:

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) \right) \eta \, d\tau = 0 \longrightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) = 0} \tag{16.46}$$

which is a natural generalization of the Euler-Lagrange equations of motion from classical mechanics.

As a check, let's see how these field equations apply to our scalar field Lagrangian $\mathcal{L} = \frac{1}{2} \phi_{,\alpha} \, g^{\alpha\beta} \, \phi_{,\beta}$ (dropping constants again):

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) &= \partial_\mu \left(g^{\mu\beta} \, \phi_{,\beta} \right) = \partial^\beta \, \partial_\beta \, \phi \\ &= -\frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 \end{aligned} \tag{16.47}$$

which is just another equivalent form for the wave equation.

So to sum up, for a metric representing $D+1$ dimensional flat space with a temporal component (signature -1 in the metric) and D Cartesian spatial components, the Euler-Lagrange equations for the Lagrange density $\mathcal{L}(\phi, \phi_{,\mu})$ corresponding to extremal action are

$$\partial_\mu \left(\frac{\partial L}{\partial \phi_{,\mu}} \right) - \frac{\partial L}{\partial \phi} = 0. \tag{16.48}$$

We could go further, this equation could be modified to handle densities like $\mathcal{L}(\phi, \phi_{,\mu}, \phi_{,\mu\nu})$ or other high-derivative combinations, but we have no reason currently to do this, just as in classical mechanics, we do not think of Lagrangians that depend on acceleration.