# Coordinate Transformations and the Action

Lecture 17

Physics 411
Classical Mechanics II

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Just as the particle Lagrangian from classical mechanics can be used to deduce properties of the system (conserved quantities, momenta), the field Lagrange densities are useful even prior to finding the field equations. In particular, the invariance of a particle action under various transformations tells us about constants of the motion via Noether's theorem: If the action is unchanged by translation, momentum is conserved, if infinitesimal rotations leave the action fixed, angular momentum is conserved, and of course, time-translation insensitivity leads to energy conservation. The same sort of ideas hold for field actions – the invariance of the action under coordinate transformation, in particular, implies that the field equations we obtain will satisfy general covariance, i.e. that the field equations will be proper tensors.

The details of integration in generic spaces take a little time to go through, and we will present a few results that will be more useful later on, but fit nicely into our current discussion. In the back of our minds is the scalar Lagrange density, a simple model to which we can apply our ideas.

## 17.1 Coordinate Transformation

There is one last wrinkle in our classical field Lagrangian. We have seen that

$$S = \int \mathcal{L}(\phi, \phi_{,\mu})\, d\tau \tag{17.1}$$

is appropriate for Cartesian spatial coordinates and time, but what if we wanted to transform to some other coordinate system?

Suppose we are in a two dimensional Euclidean space with Cartesian coordinates $x^j \doteq (x, y)^T$, so that the metric is just the identity matrix. Consider the "volume" integral, in this context:

$$I \equiv \int_{\mathcal{B}} f(x,y)\, dx dy \tag{17.2}$$

how does it change if we go to some new coordinate system, $x'^j \doteq (x', y')^T$? Well, we know from vector calculus that there is a factor of the determinant of the Jacobian of the transformation between the two coordinates involved so as to keep area elements equal. The integral written in terms of $x'$ and $y'$ is

$$I = \int_{\mathcal{B}'} f(x', y')\, |\det(J)|\, dx' dy'. \tag{17.3}$$

We can connect this to familiar objects – for a Jacobian:

$$J^j_{\ k} = J(x^j, x'^k) = \frac{\partial x^j}{\partial x'^k} \doteq \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^1}{\partial x'^2} \\ \frac{\partial x^2}{\partial x'^1} & \frac{\partial x^2}{\partial x'^2} \end{pmatrix}, \tag{17.4}$$

where we view $x^1(x'^1, x'^2)$, $x^2(x'^1, x'^2)$ as functions of the new coordinates. There is a connection between the Jacobian and the transformed metric – according to the covariant transformation law

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x_\alpha}{\partial x'^\nu} \tag{17.5}$$

using the identity-matrix form for $g_{\alpha\beta}$. This means that the new metric can be viewed as a matrix product of the Jacobian of the transformation

$$g'_{\mu\nu} = J^\alpha_{\ \mu}\, J_{\alpha\nu} \tag{17.6}$$

and the covariant $\alpha$ in the second term is irrelevant, since we start in Euclidean space. The interesting feature here is that we can relate the determinant of the Jacobian matrix to the determinant of the metric – $\det(g_{\mu\nu}) = (\det(J^\mu_{\ \nu}))^2$ (I've used the tensor notation, but we are viewing these as matrices when we take the determinant). The determinant of the metric is generally denoted $g \equiv \det(g_{\mu\nu})$ and then the integral transformation law reads

$$I' = \int_{\mathcal{B}'} f(x', y')\, \sqrt{g'}\, d\tau'. \tag{17.7}$$

### 17.1.1 Example

Take the transformation from Cartesian coordinates to circular: $x = r\cos\theta$, $y = r\sin\theta$ – the Jacobian is

$$J^j_k \doteq \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\,\sin\theta \\ \sin\theta & r\,\cos\theta \end{pmatrix} \tag{17.8}$$

and the determinant is $\det(J^j_k) = r$. The metric for circular coordinates is

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{17.9}$$

with determinant $g = r^2$, so it is indeed the case that $\det\left(J^j_k\right) = \sqrt{g}$.

If we think of the integral over a “volume” in this two-dimensional space,

$$\int dxdy = \int r\,dr\,d\phi, \tag{17.10}$$

then we can clearly see the transformation: $d\tau = dx\,dy$ is related to $d\tau' = dr\,d\phi$ via the Jacobian:

$$d\tau = |J|\,d\tau', \tag{17.11}$$

while the metric determinant in Cartesian coordinates is $g = 1$, and in polar, $g' = r^2$, so we have

$$g' = |J|^2\,g \tag{17.12}$$

and it is clear that

$$\sqrt{g}\,d\tau = \frac{|J|}{|J|}\,\sqrt{g'}\,d\tau', \tag{17.13}$$

a scalar.

### 17.1.2 Final Form of Field Action

There is only one final issue: the signature of the metrics we will be using. Since our “spaces” are really “spacetimes”, we have a Lorentzian signature metric – even with Euclidean spatial coordinates, the determinant of $g_{\mu\nu}$ for Minkowski, say, is $g = -1$, so taking the square root introduces a factor of $i = \sqrt{-1}$. To keep our volume interpretation, then, we actually have $|\det(J^\mu_\nu)| = \sqrt{-g'}$ for our spacetime metrics.

So we will introduce a factor of $\sqrt{-g}$ in all of our actions – this doesn't change anything we've done so far, since for us, $\sqrt{-g}$ has been one until now, but in the interest of generality, the action for a scalar field theory is

$$\boxed{S[\phi(x^\mu)] = \int d\tau \underbrace{\left(\sqrt{-g}\,\bar{\mathcal{L}}(\phi, \phi_{,\mu})\right)}_{\equiv \mathcal{L}},} \tag{17.14}$$

where we have a scalar $\bar{\mathcal{L}}(\phi, \phi_{,\mu})$, the Lagrange density in Euclidean spatial coordinates, and the new Lagrange density $\mathcal{L}$ is just the product of $\bar{\mathcal{L}}$ and $\sqrt{-g}$, the Jacobian factor. This is an interesting shift, because while a density like $\phi_{,\mu}\, g^{\mu\nu}\, \phi_{,\nu}$ is clearly a scalar, we suspect that $\sqrt{-g}$ is *not*, since $d\tau$ is not – the whole point of putting in the $\sqrt{-g}$ factor was to make the integration procedure itself a scalar operation.

### 17.1.3 Transformation of $g$

Let's look at how the determinant of the metric transforms under arbitrary coordinate transformation. We have:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'_\nu}\, g_{\alpha\beta} \tag{17.15}$$

and if we view the right-hand side as three matrices multiplied together, then using the multiplicative property of determinants, $\det(A\,B) = \det(A)\,\det(B)$, we have

$$g' = \det\left(\frac{\partial x}{\partial x'}\right)^2 g \tag{17.16}$$

and the matrix $\frac{\partial x}{\partial x'}$ is the Jacobian for $x' \to x$, as above. The Jacobian for the inverse transformation, taking us from $x \to x'$ is just the matrix inverse of $\frac{\partial x}{\partial x'}$, so that:

$$g' = \det\left(\frac{\partial x'}{\partial x}\right)^{-2} g \tag{17.17}$$

and a quantity that transforms in this manner, picking up a factor of the determinant of the coordinate transformation is called a tensor *density*. The name density, again, connects the role of $g$ to the transformation of volumes under integration. We say that a tensor density has weight given by $p$ in the general transformation:

$$A' = \det\left(\frac{\partial x'}{\partial x}\right)^p A \tag{17.18}$$

(for scalar densities) – so the determinant of the metric is a scalar density of weight $p = -2$, and the square root is a scalar density of weight $-1$. The volume factor itself:

$$d\tau' = \det\left(\frac{\partial x'}{\partial x}\right) d\tau \tag{17.19}$$

is a scalar density of weight $p = 1$. Density weights add when multiplied together. For example:

$$\sqrt{-g'}\, d\tau' = \left(\sqrt{-g}\, \det\left(\frac{\partial x'}{\partial x}\right)^{-1}\right)\left(\det\left(\frac{\partial x'}{\partial x}\right) d\tau\right) = \sqrt{-g}\, d\tau \tag{17.20}$$

and we see that $\sqrt{-g}\, d\tau$ is a density of weight 0, which is to say, a normal scalar.

Again, if we take our usual $\bar{\mathcal{L}}$ to be a scalar (like $\phi_{,\mu}\, g^{\mu\nu}\, \phi_{,\nu}$), then $\mathcal{L} = \bar{\mathcal{L}}\sqrt{-g}$ is a scalar density of weight $-1$, just right for forming an action.

### 17.1.4 Tensor Density Derivatives

While we're at it, it's a good idea to set some of the notation for derivatives of densities, as these come up any time integration is involved. Recall the covariant derivative of a first rank (zero-weight) tensor:

$$A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu{}_{\sigma\nu}\, A^\sigma. \tag{17.21}$$

What if we had a tensor density of weight $p$: $\mathcal{A}^\mu$? We can construct a true tensor $A^\mu = (\sqrt{-g})^p\, \mathcal{A}^\mu$ from this, then apply the covariant derivative as above, and finally multiply by $\sqrt{-g}^{-p}$ to restore the proper weight – this suggests that

$$\begin{aligned}
\mathcal{A}^\mu{}_{;\nu} &= (\sqrt{-g})^{-p}\left((\sqrt{-g})^p\, \mathcal{A}^\mu\right)_{;\nu} \\
&= (\sqrt{-g})^{-p}\left(((\sqrt{-g})^p\, \mathcal{A}^\mu)_{,\nu} + \Gamma^\mu{}_{\sigma\nu}\,(\sqrt{-g})^p\, \mathcal{A}^\sigma\right) \\
&= \mathcal{A}^\mu{}_{,\nu} + \Gamma^\mu{}_{\sigma\nu}\, \mathcal{A}^\sigma + \mathcal{A}^\mu\, \frac{p}{2\, g}\, \frac{\partial g}{\partial x^\nu}.
\end{aligned} \tag{17.22}$$

The advantage of $\sqrt{-g}$ in integration comes from the observation that, for a tensor $A^{\mu}$[1]:

$$\boxed{\sqrt{-g}\,A^{\mu}{}_{;\mu} = \left(\sqrt{-g}\,A^{\mu}\right)_{,\mu}}\,, \tag{17.23}$$

so we can express the Cartesian divergence theorem: $\int_V \nabla\cdot\mathbf{E}\,d\tau = \oint_{\partial V}\mathbf{E}\cdot d\mathbf{a}$ in multi-dimensional form:

$$\int d\tau\,\sqrt{-g}\,A^{\mu}{}_{;\mu} = \oint \sqrt{-g}\,A^{\mu}\,da_{\mu}. \tag{17.24}$$

Notice also that the covariant divergence of the integrand reduces to the "ordinary" derivative when $\sqrt{-g}$ is involved: $(\sqrt{-g}\,A^{\mu})_{;\mu} = (\sqrt{-g}\,A^{\mu})_{,\mu}$.

## 17.2 Scalar Fields in Spherical Coordinates

Let's see how all of this works for our scalar field. We have the general field equations

$$\left(\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) - \frac{\partial\mathcal{L}}{\partial\phi}\right) = 0, \tag{17.25}$$

and if we take the usual four-dimensional $(ct, x, y, z)$ flat coordinates, then

$$\mathcal{L} = \frac{1}{2}\,\phi_{,\mu}\,g^{\mu\nu}\,\phi_{,\nu}\,\sqrt{-g} = \frac{1}{2}\,\phi_{,\mu}\,g^{\mu\nu}\,\phi_{,\nu} \tag{17.26}$$

and we recover

$$\partial_{\mu}\,\partial^{\mu}\phi = -\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = 0. \tag{17.27}$$

But suppose we have in mind spherical coordinates? Then

$$g_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\sin^2\theta \end{pmatrix} \tag{17.28}$$

and $\sqrt{-g} = r^2\,\sin\theta$. We have

$$\partial_{\mu}\left(g^{\mu\beta}\,\sqrt{-g}\,\phi_{,\beta}\right) = 0. \tag{17.29}$$

[1] This relies on the two properties of derivatives: $\sqrt{-g}_{;\mu} = 0$ which is pretty clear via the chain rule, and $A^{\mu}{}_{;\mu} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,A^{\mu}\right)_{,\mu}$.

Separating out the temporal and spatial portions, this gives

$$-\frac{1}{c^2}\frac{\partial^2 \phi}{dt^2} + \nabla^2 \phi = 0 \tag{17.30}$$

where now the field $\phi = \phi(t, r, \theta, \phi)$ and the Laplacian refers to the spherical one.

This is an interesting development – we started with a Lagrange density that made the action a scalar, and now we discover that the field equations themselves behave as tensors. Appropriately, the field equations have the same form no matter what coordinate system we use!

## 17.3 Coordinate Invariance

The coordinate invariance we have just seen follows directly from the action itself – since we have been careful to write the action as a scalar, it does not change under a change in coordinates. We can take $x^\mu \to x'^\mu$ and $S$ retains its numerical value (we integrate over all of the coordinates, with integration region shifted appropriately). That sounds like a transformation that should lead to some sort of conserved quantity. Next time, we will carry out the transformation explicitly, but the basic idea is to perform the coordinate transformation, which induces some transformation in the fields and their derivatives – viewed as an expansion, this gives us a perturbation to $S$ which then must be zero, enforcing the invariance of the action.