Vector Field Theory (E&M)

Lecture 21

Physics 411
Classical Mechanics II

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We now move from first-order scalar field Lagrange densities to the equivalent form for a vector field. Our model (and ultimate goal) is a description of vacuum electrodynamics. From the natural action, we derive the field equations – Maxwell's equations for potentials in the absence of sources. Sourceless field theories tell us nothing about the various constants which must appear in a general action, coupling sources to the field theory density, but by appealing to our E&M background, we can set the various constants appropriate to source coupling.

So we begin with the action and field equations, then form the energy-momentum tensor for the fields alone, and by setting our one overall constant so that the T^{00} component corresponds to energy density, we recover the full (familiar) energy-momentum tensor for E&M. After all of this, we are in good position to introduce charge and current sources.

21.1 Vector Action

Our objective is E&M, and our starting point will be an action. The big change comes in our Lagrange density – rather than some simple $\mathcal{L}(\phi, \phi_{,\mu}, g_{\mu\nu})$, we have a set of fields, call them A_μ, so the action can depend on scalars made out of the fields A_μ and their derivatives $A_{\mu,\nu}$. Think of Minkowski space, and our usual treatment of electricity and magnetism. We know that the "Field strength tensor" $F^{\mu\nu}$ is a Lorentz-tensor, but why stop there? Why not make a full scalar action out of the field-strength tensor, and ignore its usual definition (we should recover this anyway, if our theory is properly defined)? To that end, the only important feature of $F^{\mu\nu}$ is its antisymmetry: $F^{\mu\nu} = -F^{\nu\mu}$.

Since we are shooting for a vector field, a singly-indexed set of quantities, the second rank tensor $F^{\mu\nu}$ *must* be associated, in first order form, with the momenta. The only scalar we can make comes from double-contraction $F^{\mu\nu}\,F_{\mu\nu}$. So taking this as the "Hamiltonian", our first-order action has to look like

$$S_V = \int d\tau\,\sqrt{-g}\left(F^{\mu\nu}\,(\text{derivatives of } A_\mu) - \frac{1}{2}\,F^{\mu\nu}\,g_{\alpha\mu}\,g_{\beta\nu}\,F^{\alpha\beta}\right). \tag{21.1}$$

Because $F^{\mu\nu}$ is antisymmetric, we can form an antisymmetrization of $A_{\nu,\mu}$, i.e. $(A_{\nu,\mu} - A_{\mu,\nu})$. As an interesting aside, we are implicitly assuming that the metric here is just the usual Minkowski, $g_{\mu\nu} = \eta_{\mu\nu}$, so that one might object to the normal partial derivatives on the $A_{\mu,\nu}$. This is actually not a problem if we use the antisymmetrized form, since

$$\begin{aligned} A_{\nu;\mu} - A_{\mu;\nu} &= \left(A_{\nu,\mu} - \Gamma^\sigma{}_{\nu\mu}\,A_\sigma\right) - \left(A_{\mu,\nu} - \Gamma^\sigma{}_{\mu\nu}\,A_\sigma\right) \\ &= A_{\nu,\mu} - A_{\mu,\nu}, \end{aligned} \tag{21.2}$$

using the symmetry (torsion-free) of the connection: $\Gamma^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\nu\mu}$.

So our proposed action is

$$\boxed{S_V = \int d\tau\,\sqrt{-g}\left(F^{\mu\nu}\,(A_{\nu,\mu} - A_{\mu,\nu}) - \frac{1}{2}\,F^{\mu\nu}\,g_{\alpha\mu}\,g_{\beta\nu}\,F^{\alpha\beta}\right).} \tag{21.3}$$

Again, since this is first-order form, we vary w.r.t. $F^{\mu\nu}$ and A_μ separately. Incidentally, the antisymmetry of $F^{\mu\nu}$ is now enforced – because it is contracted with the antisymmetrization of $A_{\mu,\nu}$, only the antisymmetric portion will contribute, so we can vary w.r.t. all sixteen components independently.

$$\frac{\delta S_V}{\delta F^{\mu\nu}} = (A_{\nu,\mu} - A_{\mu,\nu}) - g_{\alpha\mu} g_{\beta\nu}\,F^{\alpha\beta} = 0 \tag{21.4}$$

and this gives a relation between F and ∂A:

$$\boxed{F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}.} \tag{21.5}$$

The variation of the A_μ is a little more difficult – with an obvious abuse of notation:

$$\begin{aligned} \delta S_V &= \int d\tau\,\sqrt{-g}\,\frac{\partial}{\partial A_{\mu,\nu}}\left(F^{\mu\nu}\,(A_{\nu,\mu} - A_{\mu,\nu})\right)\,\delta A_{\mu,\nu} \\ &= \int d\tau\,\sqrt{-g}\,\frac{\partial}{\partial A_\mu}\left(A_{\mu,\nu}\,(F^{\nu\mu} - F^{\mu\nu})\right)\,\delta A_{\mu,\nu} \\ &= \int d\tau\,\sqrt{-g}\,(F^{\nu\mu} - F^{\mu\nu})\,\delta A_{\mu,\nu}, \end{aligned} \tag{21.6}$$

and we really have in mind the covariant derivative (the $\sqrt{-g}$ ensures that we mean what we say).

We can use integration by parts to get a total divergence term, which as usual can be converted to a boundary integral where it must vanish,

$$\delta S_V = \underbrace{\int d\tau \left[\sqrt{-g}\,(F^{\nu\mu} - F^{\mu\nu})\,\delta A_\mu\right]_{;\nu}}_{=0} - \int d\tau \left[\sqrt{-g}\,(F^{\nu\mu} - F^{\mu\nu})\right]_{;\nu}\,\delta A_\mu. \tag{21.7}$$

The second term's integrand must then vanish for arbitrary δA_μ. Using $(\sqrt{-g})_{;\nu} = 0$, the above gives the second set of field equations:

$$(F^{\nu\mu} - F^{\mu\nu})_{;\nu} = 0 \tag{21.8}$$

and because we know that $F_{\mu\nu}$ is antisymmetric (either *a priori* or from (21.5)), this reduces to

$$\boxed{F^{\mu\nu}{}_{;\nu} = 0.} \tag{21.9}$$

Combining these two, and writing in terms of A_μ, we find that the fields A_μ satisfy

$$\partial^\mu\,\partial_\nu\,A^\nu - \partial^\nu\,\partial_\nu\,A^\mu = 0. \tag{21.10}$$

If we are really in Minkowski space with Cartesian spatial coordinates, this is a set of four independent equations which we may write as

$$\begin{aligned} 0 &= -\frac{1}{c}\,\frac{\partial}{\partial t}\left[\frac{1}{c}\,\frac{\partial A^0}{\partial t} + \nabla\cdot\mathbf{A}\right] - \left[-\frac{1}{c^2}\,\frac{\partial^2 A^0}{\partial t^2} + \nabla^2 A^0\right] \\ 0 &= \nabla\left[\frac{1}{c}\,\frac{\partial A^0}{\partial t} + \nabla\cdot\mathbf{A}\right] - \left[-\frac{1}{c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2} + \nabla^2\mathbf{A}\right]. \end{aligned} \tag{21.11}$$

Note the similarity with equations (10.4) and (10.5) from Griffiths[1]:

$$\begin{gathered} \nabla^2 V + \frac{\partial}{\partial t}\,(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\epsilon_0} \\ \left(\nabla^2\mathbf{A} - \mu_0\,\epsilon_0\,\frac{\partial^2\mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\,\epsilon_0\,\frac{\partial V}{\partial t}\right) = -\mu_0\,\mathbf{J}. \end{gathered} \tag{21.12}$$

It appears reasonable to interpret $A^0 = \frac{V}{c}$ and $A^j = \mathbf{A}$, the electric and magnetic vector potentials respectively. To proceed, we know that there

[1] Griffiths, David J. *Introduction to Electrodynamics.* Prentice Hall, 1999. See page 417 of the third edition.

must be gauge freedom – the physical effects of the fields are transmitted through $F^{\mu\nu}$. We know this to be true, although we have not yet done anything to establish it here, there being no sources in our theory. The gauge freedom is expressed in the connection between the field strength tensor and the potential

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{21.13}$$

If we take $A_\mu \to A'_\mu = A_\mu + \psi_{,\mu}$, then

$$F'_{\mu\nu} = A'_{\nu,\mu} - A'_{\mu,\nu} = (A_{\nu,\mu} + \psi_{,\nu\mu}) - (A_{\mu,\nu} + \psi_{,\mu\nu}) = A_{\nu,\mu} - A_{\mu,\nu}, \tag{21.14}$$

and nothing changes. We can exploit this to set the total divergence of A^μ. Suppose we start with a set A_μ such that $\partial_\mu A^\mu = F(x)$, then we introduce a four-gradient: $A^\mu + \partial^\mu \psi$, the new divergence is

$$\partial_\mu \left(A^\mu + \partial^\mu \psi\right) = F + \partial_\mu \partial^\mu \psi \tag{21.15}$$

and if we like, this new divergence can be set to zero by appropriate choice of ψ (i.e. $\partial_\mu \partial^\mu \psi = -F$, Poisson's equation for the D'Alembertian operator). Then we can *a priori* set $\partial_\mu A^\mu = 0$ and the field equations can be simplified (from (21.10))

$$\partial^\mu \partial_\nu A^\nu - \partial^\nu \partial_\nu A^\mu = -\partial^\nu \partial_\nu A^\mu \tag{21.16}$$

which becomes, under our identification $A^0 = \frac{V}{c}$:

$$\begin{aligned} 0 &= -\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + \nabla^2 V \\ 0 &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A}, \end{aligned} \tag{21.17}$$

appropriate to source-free potential formulation of E&M in Lorentz gauge.

21.1.1 The Field Strength Tensor

Now we can go on to define the components of the field strength tensor, but we are still lacking an interpretation in terms of **E** and **B**. That's fine, for now, we can use our previous experience as a crutch, but to reiterate – we cannot say exactly what the fields are (the independent elements of $F^{\mu\nu}$) until we have a notion of force.

From $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$, we have[2]

$$F^{\mu\nu} \doteq \begin{pmatrix} 0 & -\frac{1}{c}\left(\frac{\partial A^x}{\partial t} + \frac{\partial V}{\partial x}\right) & -\frac{1}{c}\left(\frac{\partial A^y}{\partial t} + \frac{\partial V}{\partial y}\right) & -\frac{1}{c}\left(\frac{\partial A^z}{\partial t} + \frac{\partial V}{\partial z}\right) \\ \circ & 0 & \frac{\partial A^y}{\partial x} - \frac{\partial A^x}{\partial y} & \frac{\partial A^z}{\partial x} - \frac{\partial A^x}{\partial z} \\ \circ & \circ & 0 & \frac{\partial A^z}{\partial y} - \frac{\partial A^y}{\partial z} \\ \circ & \circ & \circ & 0 \end{pmatrix}, \tag{21.18}$$

with $\circ$ representing the obvious antisymmetric entry.

If we take the typical definition $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, then the field strength tensor takes its usual form

$$\boxed{F^{\mu\nu} \doteq \begin{pmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B^z & -B^y \\ -\frac{E^y}{c} & -B^z & 0 & B^x \\ -\frac{E^z}{c} & B^y & -B^x & 0 \end{pmatrix}.} \tag{21.19}$$

21.2 Energy-Momentum Tensor for E&M

We now understand the Lagrangian density (21.3) as the appropriate integrand of the action for the electromagnetic potential, or at least, the source-free E&M form. Our next immediate task is to identify the energy-momentum tensor for this field theory, and we will do this via the definition of the $T^{\mu\nu}$ tensor:

$$T^{\mu\nu} = -\left(g^{\mu\nu}\,\bar{\mathcal{L}} + 2\,\frac{\partial \bar{\mathcal{L}}}{\partial g_{\mu\nu}}\right) \tag{21.20}$$

with

$$\bar{\mathcal{L}} = F^{\mu\nu}\left(A_{\nu,\mu} - A_{\mu,\nu}\right) - \frac{1}{2}\,F^{\sigma\rho}\,g_{\alpha\sigma}\,g_{\beta\rho}\,F^{\alpha\beta}. \tag{21.21}$$

The most important term is the derivative w.r.t. $g_{\mu\nu}$, appearing in the second term above – we can simplify life by rewriting one set of dummy indices so as to get a $g_{\mu\nu}$ in between the two field strengths. Remember that we could do this on either of the two metrics, so we will pick up an overall factor of

[2] again using Minkowski, the "usual" gradient is a covariant tensor, while the four-potential A^μ is naturally contravariant, so it is A^μ that is represented by $(\frac{V}{c}, A^x, A^y, A^z)$ for example. Raising and lowering changes the sign of the zero component only.

two:

$$\begin{aligned} \frac{\partial}{\partial g_{\mu\nu}} \left(\frac{1}{2} F^{\sigma\rho} \, g_{\alpha\sigma} \, g_{\beta\rho} \, F^{\alpha\beta} \right) &= \frac{\partial}{\partial g_{\mu\nu}} \left(\frac{1}{2} F^{\nu\rho} \, g_{\mu\nu} \, g_{\beta\rho} \, F^{\mu\beta} \right) \\ &= F^{\nu\rho} \, g_{\beta\rho} \, F^{\mu\beta}. \end{aligned} \tag{21.22}$$

Putting this together with the field relation between $F^{\mu\nu}$ and the derivatives of A_μ, we have

$$\begin{aligned} T^{\mu\nu} &= -\left(g^{\mu\nu} \left(\frac{1}{2} F^{\alpha\beta} \, F_{\alpha\beta} \right) - 2 \, F^{\nu\rho} \, g_{\beta\rho} \, F^{\mu\beta} \right) \\ &= 2 \left(F^{\mu\beta} \, F^{\nu}{}_{\beta} - \frac{1}{4} \, g^{\mu\nu} \, F^{\alpha\beta} \, F_{\alpha\beta} \right). \end{aligned} \tag{21.23}$$

Overall factors in a free Lagrange density do not matter (field equations equal zero, so there is no need to worry about constants). For coupling to matter, however, we must put in the appropriate units. One way to do this is to connect the stress tensor to its known form. Suppose we started from the action $\tilde{S}_V \equiv \alpha \, S_V$, then the energy-momentum tensor is

$$\boxed{T^{\mu\nu} = 2 \, \alpha \left(F^{\mu\beta} \, F^{\nu}{}_{\beta} - \frac{1}{4} \, g^{\mu\nu} \, F^{\alpha\beta} \, F_{\alpha\beta} \right).} \tag{21.24}$$

Take the zero component of this,

$$T^{00} = \alpha \left(\mathbf{B} \cdot \mathbf{B} + \frac{1}{c^2} \, \mathbf{E} \cdot \mathbf{E} \right). \tag{21.25}$$

The usual expression for energy density is given by (Griffiths (8.32), for example):

$$u = \frac{1}{2} \left(\epsilon_0 \, E^2 + \frac{1}{\mu_0} \, B^2 \right) = \frac{\epsilon_0}{2} \left(E^2 + c^2 \, B^2 \right) \tag{21.26}$$

so evidently, we can identify

$$T^{00} = \frac{2 \, \alpha}{\epsilon_0 \, c^2} \, u \longrightarrow T^{00} = u \quad \text{for } \alpha = \tfrac{\epsilon_0 \, c^2}{2}. \tag{21.27}$$

With this factor in place, we have the final form

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2} \, \epsilon_0 \left(E^2 + c^2 \, B^2 \right) & c \, \epsilon_0 \, \mathbf{E} \times \mathbf{B} \\ c \, \epsilon_0 \, \mathbf{E} \times \mathbf{B} & -T^{ij} \end{pmatrix}, \tag{21.28}$$

with T^{ij} the usual Maxwell stress tensor:

$$T^{ij} = \epsilon_0 \left(E^i \, E^j - \frac{1}{2} \, \delta^{ij} \, E^2 \right) + \epsilon_0 \, c^2 \left(B^i \, B^j - \frac{1}{2} \, \delta^{ij} \, B^2 \right). \tag{21.29}$$

We have the association $\frac{1}{c} \, T^{0j} = \epsilon_0 \, \mathbf{E} \times \mathbf{B}$, the familiar momentum density from E&M. The energy density is indeed the T^{00} component, and the spatial portion is (the negative of) Maxwell's stress tensor.

21.2.1 Units

The above allowed us to set SI units for the action via the prefactor $\frac{\epsilon_0 \, c^2}{2} = \frac{1}{2 \, \mu_0}$, the electromagnetic action reads:

$$S_V = \frac{1}{2 \, \mu_0} \int d\tau \, \sqrt{-g} \left(F^{\mu\nu} \left(A_{\nu,\mu} - A_{\mu,\nu} \right) - \frac{1}{2} \, F^{\mu\nu} \, g_{\alpha\mu} \, g_{\beta\nu} \, F^{\alpha\beta} \right), \tag{21.30}$$

but this is not the most obvious, certainly not the most usual, form for the action formulation of E&M. Most common are gaussian units, where E and B have the same units. The basic rule taking us from SI to gaussian is: $\epsilon_0 \longrightarrow \frac{1}{4\,\pi}$, $\mu_0 \longrightarrow \frac{4\,\pi}{c^2}$ (ensuring that $\frac{1}{\epsilon_0 \, \mu_0} = c^2$), and $c\mathbf{B} \longrightarrow \mathbf{B}$. For example, the electromagnetic field energy in a volume in SI units is:

$$U = \frac{1}{2} \int \left(\epsilon_0 \, E_{SI}^2 + \frac{1}{\mu_0} \, B_{SI}^2 \right) \, d\tau \tag{21.31}$$

using our conversion, we have

$$\begin{aligned} U &= \frac{1}{2} \int \left(\frac{1}{4\,\pi} \, E^2 + \frac{c^2}{\mu_0} \, B_{SI}^2 \right) \, d\tau \\ &= \frac{1}{8\,\pi} \int \left(E^2 + B^2 \right) \, d\tau. \end{aligned} \tag{21.32}$$

We can convert the action itself – in its usual form, we write the action entirely in terms of the field strength tensor $F^{\mu\nu}$, so we lose the connection between the potential and the fields.

$$S_V = \frac{1}{16\,\pi} \int d\tau \, F^{\mu\nu} F_{\mu\nu}, \tag{21.33}$$

where the c^2 that was in the numerator from $\mu_0 \longrightarrow \frac{4\,\pi}{c^2}$ got soaked into the field strength tensor – because it is quadratic, the integrand depends on components like $\sim \frac{E^i E^j}{c^2}$ and the c^2 out front kills that, and in addition, $\sim B^i_{SI} B^j_{SI}$ which becomes, upon multiplication by c^2, just $\sim B^i B^j$. In other words, the field strength tensor in gaussian units looks like (21.19) with $c \longrightarrow 1$. As a final note, you will generally see this action written with an overall minus sign – that is convention, and leads to a compensating sign in the stress tensor definition – regardless, keep the physics in mind, and you will be safe.

21.3 Conservation of Angular Momentum

Given the conservation of the energy-momentum tensor itself, we can also construct the angular momentum analogue – define:

$$M^{\alpha\beta\gamma} \equiv x^\beta T^{\gamma\alpha} - x^\gamma T^{\beta\alpha}, \tag{21.34}$$

then we have the derivative:

$$\frac{\partial M^{\alpha\beta\gamma}}{\partial x^\alpha} = T^{\gamma\beta} - T^{\beta\gamma} = 0, \tag{21.35}$$

from the symmetry of the stress tensor. This gives us a set of conserved quantities in the usual way:

$$\frac{1}{c}\frac{\partial M^{0\beta\gamma}}{\partial t} = -\frac{\partial M^{i\beta\gamma}}{\partial x^i} \tag{21.36}$$

or, in integral form:

$$\frac{d}{dt}\int \frac{1}{c} M^{0\beta\gamma}\, d\tau = -\oint M^{i\beta\gamma}\, da_i. \tag{21.37}$$

The left-hand side represents six independent quantities – of most interest are the spatial components – let $J^{\alpha\beta} = M^{0\beta\gamma}$, then the densities are just as we expect:

$$J^{ij} = M^{0ij} = x^i T^{j0} - x^j T^{i0}, \tag{21.38}$$

so that, in terms of the spatial components of momentum, we have:

$$J^{ij} = x^i p^j - x^j p^i, \tag{21.39}$$

precisely the angular momentum.