E&M and Sources

Lecture 22

Physics 411
Classical Mechanics II

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E&M is a good place to begin talking about sources, since we already know the answer from Maxwell's equations. We look at how a source J^μ can be correctly coupled to a free vector field. By introducing the simplest possible term in the action, it is clear that the coupling is correct, but will lead to inconsistencies since there is no field theory for J^μ – we imagine a set of charges and currents specified in the usual way, and the action will be incomplete until the terms associated with the free field theory for J^μ are introduced.

In addition to giving us a model for how to couple to a field theory, we expose a new symmetry of the action in the new source term. The gauge invariance of $F^{\mu\nu}$ (changing the four-potential A^μ by a total derivative) implies that J^μ, whatever it may be, is conserved. This leads to an interest in more general J^μ, not just those coming from macroscopic E&M. So we begin our discussion of coupling *fields* to E&M, and for this, we need some massive scalar field results.

22.1 Introducing Sources

We must now introduce some notion of source for the electromagnetic field – we are looking for a connection between source charges ρ and currents, and the electric and magnetic fields themselves. In this setting, the target is a relation between $F^{\mu\nu}$ (or A^μ) and $(\rho, \mathbf{J})$. We have not yet attempted to introduce sources, and indeed it is unclear in general how sources should relate to $F^{\mu\nu}$. Fortunately, we are familiar with the structure of electrodynamics, and can use this to guide our approach.

For the four-potential, in Lorentz gauge, we know that Maxwell's equations reduce to

$$
\begin{aligned}
-\frac{\rho}{\epsilon_0} &= \nabla^2 V - \frac{1}{c^2}\,\frac{\partial^2 V}{\partial t^2} \\
-\mu_0\,\mathbf{J} &= \nabla^2 \mathbf{A} - \frac{1}{c^2}\,\frac{\partial^2 \mathbf{A}}{\partial t^2},
\end{aligned}
\tag{22.1}
$$

or, in four-vector language (with $A^0 = \frac{V}{c}$),

$$
\left(-\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} + \nabla^2\right)\left(\begin{array}{c}\frac{V}{c}\\ \mathbf{A}\end{array}\right) = -\left(\begin{array}{c}\frac{\rho\,c}{\epsilon_0\,c^2}\\ \frac{\mathbf{J}}{\epsilon_0\,c^2}\end{array}\right),
\tag{22.2}
$$

which implies the usual identification:

$$
J^\mu \doteq \left(\begin{array}{c}\rho\,c\\ \mathbf{J}\end{array}\right)
\tag{22.3}
$$

and the equations become

$$
\partial^\nu\,\partial_\nu A^\mu = -\frac{1}{\epsilon_0\,c^2}\,J^\mu.
\tag{22.4}
$$

These are to be introduced via the $F^{\mu\nu}$ equation – we have

$$
\partial_\nu F^{\mu\nu} = \partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \underbrace{= -\partial_\nu\,\partial^\nu A^\mu}_{\text{in Lorentz gauge}} = \frac{1}{\epsilon_0\,c^2}\,J^\mu.
\tag{22.5}
$$

Remember that the above equation $\partial_\nu F^{\mu\nu} = 0$, in the source-free case, came from variation of A^μ. Our goal, then, is to introduce a term in the Lagrangian that gives, upon variation of A^μ, the source $\frac{1}{\epsilon_0\,c^2}\,J^\mu$. The most obvious such term (that is still a scalar) is $A_\mu\,J^\mu$. Suppose we make the natural generalization

$$
S = \frac{\epsilon_0\,c^2}{2}\int d\tau\,\sqrt{-g}\left(F^{\mu\nu}\left(A_{\nu,\mu} - A_{\mu,\nu}\right) - \frac{1}{2}\,F^{\mu\nu}\,F_{\mu\nu}\right) + \beta\int d\tau\,\sqrt{-g}\,J^\mu\,A_\mu,
\tag{22.6}
$$

then variation of $F^{\mu\nu}$ gives, as before:

$$
\frac{\epsilon_0\,c^2}{2}\left(A_{\nu,\mu} - A_{\mu,\nu} - F_{\mu\nu}\right) = 0
\tag{22.7}
$$

and we recover the description of field strength in terms of the potential fields. The variation w.r.t. A_μ is different – we get

$$
\int d\tau\,\sqrt{-g}\left[\frac{\epsilon_0\,c^2}{2}\left(F^{\nu\mu} - F^{\mu\nu}\right)_{;\mu} + \beta\,J^\nu\right]\delta A_\mu = 0,
\tag{22.8}
$$

and using the antisymmetry of $F^{\mu\nu}$, this gives us the field equation

$$\epsilon_0\, c^2\, F^{\nu\mu}{}_{;\mu} = -\beta\, J^\nu \tag{22.9}$$

which is supposed to read: $F^{\nu\mu}{}_{;\mu} = \mu_0\, J^\nu \longrightarrow \beta = -1$.

So the coupling of E&M to sources gives an action of the form

$$S = \frac{\epsilon_0\, c^2}{2} \int d\tau\, \sqrt{-g} \left(F^{\mu\nu}\,(A_{\nu,\mu} - A_{\mu,\nu}) - \frac{1}{2}\, F^{\mu\nu}\, F_{\mu\nu} - \frac{2}{\epsilon_0\, c^2}\, J^\mu\, A_\mu \right). \tag{22.10}$$

Notice that, from the field equation itself: $F^{\mu\nu}{}_{;\nu} = \mu_0\, J^\mu$, we have

$$F^{\mu\nu}{}_{;\nu\mu} = \mu_0\, J^\mu{}_{;\mu} = 0 \tag{22.11}$$

because we are contracting an antisymmetric tensor $F^{\mu\nu}$ with the symmetric $\partial_\mu\partial_\nu$. This enforces charge conservation:

$$J^\mu{}_{;\mu} = \frac{1}{c}\, \frac{\partial(c\,\rho)}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{22.12}$$

We have defined J^μ in its usual E&M way, but that means that J^μ is itself a field. Where are the terms corresponding to the dynamic properties of J^μ? Presumabely, this four-current is generated by some distribution of mass and current, specified by ρ and $\mathbf{v}$. It's all true, of course, and what we are apparently lacking is an additional term in the action corresponding to the free-field action for J^μ.

22.2 Kernel of Variation

Irrespective of the lack of a complete description of the system $(J^\mu, F^{\mu\nu})$, we nevertheless can make some progress just by knowing the form of the source. That there should be *some* source and that it should be describable by J^μ is already a new situation.

This new $A_\mu J^\mu$ term provides a new type of symmetry. We know that generic variation of $A_\mu \longrightarrow A_\mu + \delta A_\mu$ leads to the equations of motion. But the free field action for E&M is actually less restrictive than this – consider the explicit variation:

$$\begin{aligned} S[A_\mu + \delta A_\mu] &= \int d\tau\, \sqrt{-g} \left(F^{\mu\nu}\,(A_{\nu,\mu} - A_{\mu,\nu}) - \frac{1}{2}\, F^{\mu\nu}\, F_{\mu\nu} \right) \\ &\quad + \int d\tau\, \sqrt{-g}\, \left(F^{\mu\nu}\,(\delta A_{\nu,\mu} - \delta A_{\mu,\nu})\right) \end{aligned} \tag{22.13}$$

so we have

$$\delta S = \int d\tau \sqrt{-g} \left(F^{\mu\nu} - F^{\nu\mu} \right) \delta A_{\nu,\mu} \tag{22.14}$$

and it was the integration-by-parts that gave us the statement: $F^{\mu\nu}_{\ \ \ ;\nu} = 0$. But we see here that there is an entire class of possibilites that are completely missed in the above – forgetting for the moment that $F^{\mu\nu}$ is anti-symmetric, it certainly appears in anti-symmetric form here, and so if it were the case that $\delta A_{\nu,\mu} = \delta A_{\mu,\nu}$, then $\delta S = 0$ automatically and we have not constrained the fields at all! So while it is true that the integration-by-parts argument will work for arbitrary variation of A_μ, we have apparently missed the class: $\delta A_{\nu,\mu} = \eta_{,\nu\mu}$ for arbitrary scalar functions, entirely. Well, to say that they are "missing" is a little much, rather, they don't matter. The dynamical fields are left unchanged by a transformation of A_μ of the form:

$$A_\mu \longrightarrow A'_\mu = A_\mu + \eta_{,\mu} \tag{22.15}$$

in which case $\delta S = 0$ anyway. This is precisely the gauge choice that underlies E&M. Note the similarity with the other "choice leads to conservation" argument we have seen recently – that of coordinate invariance in an action and energy-momentum tensor conservation. It's a similar situation here with the gauge choice for potentials, and in GR, we refer to coordinate choice as a gauge choice.

What is interesting is that when we imagine a source of the form $A_\mu J^\mu$, there enters a naked δA_μ, and so it is no longer the case that we can free ourselves of a potential restriction. In other words, for all variations δA_μ *but* $\delta A_\mu = \eta_{,\mu}$, we know the field equations. But in that special case, we have a contribution that comes from

$$\delta S_J = \int d\tau \sqrt{-g}\, J^\mu \,\delta A_\mu = \int d\tau \sqrt{-g}\, J^\mu \eta_{,\mu} \tag{22.16}$$

and we must have this term vanish – integrating by parts gives

$$\delta S_J = \underbrace{\int d\tau \left(\sqrt{-g}\, J^\mu \eta \right)_{,\mu}}_{=0} - \int d\tau \sqrt{-g}\, J^\mu_{\ ;\mu}\, \eta \tag{22.17}$$

and we see that $J^\mu_{\ ;\mu} = 0$ is a *requirement* if we are to obtain $\delta S_J = 0$. Then gauge freedom has actually imposed continuity on the sources, apart from our original E&M concerns. This tells us, in particular, that if we were looking for more exotic sources, they would all need a notion of continuity, or would violate the most minimal coupling.

Evidently, this particular parametrization of the variation is, in a sense, in the null space of the variation of the action, once we have specified the subtractive (antisymmetric) term, we automatically miss any variation of this form – with the additional structure of the current-vector coupling, we have revealed the deficiency. Now let's go back a few steps and see how this works in our earlier theories.

22.3 Scalar Fields

Recall the free-field Lagrangian for a (massless) scalar field

$$S = \int d\tau \sqrt{-g} \left(\phi_{,\mu} \pi^\mu - \frac{1}{2} \pi_\mu \pi^\mu \right). \tag{22.18}$$

When we vary w.r.t. π^μ, we find $\phi_{,\mu} = \pi_\mu$, which is reasonable. But think about the $\phi_{,\mu}$ variation. As with E&M, only the derivatives of ϕ show up, so we expect trouble. Indeed, for generic $\delta\phi$:

$$\delta S = \int d\tau \sqrt{-g} \, \pi^\mu \, \delta\phi_{,\mu} \tag{22.19}$$

and once again, we see that for variation of the form $\delta\phi$ =const. the action automatically vanishes, for all the rest, we obtain $\pi^\mu{}_{;\mu} = 0$. This is the usual sort of idea for scalars, that fundamentally, a constant can be added to the definition without changing the equations of motion.

What if we introduce a potential for ϕ – there are a few terms we might consider, $\sim \phi$, $\sim \phi^2$, etc. The simplest would be $\alpha \, \phi$, but this gives a relatively uninteresting Poisson equation with constant source. A term quadratic in ϕ changes the differential equation. For the general case,

$$S_m = \int d\tau \sqrt{-g} \left(\pi^\alpha \, \phi_{,\alpha} - \frac{1}{2} \pi^\alpha \pi^\beta g_{\alpha\beta} - V(\phi) \right) \tag{22.20}$$

we have, upon variation of π^α, the usual $\pi^\beta g_{\alpha\beta} = \phi_{,\alpha}$, and under $\delta\phi$:

$$\delta S_m = \int d\tau \sqrt{-g} \left(-\pi^\alpha{}_{;\alpha} - \frac{\partial V}{\partial \phi} \right) \delta\phi = 0 \tag{22.21}$$

which, in flat Minkowksi spacetime becomes:

$$\Box^2 \phi + \frac{\partial V}{\partial \phi} = 0. \tag{22.22}$$

For a quadratic "mass" term, $V(\phi) = -\frac{1}{2}\, m^2\, \phi^2$, we get the Klein-Gordon equation

$$\Box^2 \phi - m^2\, \phi = 0. \tag{22.23}$$