Second Rank Tensor Field Theory

Lecture 25

Physics 411
Classical Mechanics II

October 31st, 2007

We have done the warm-up: E&M. The first rank tensor field theory we developed last time gives us the guiding principles for generating higher-rank theories. Second rank tensors present a proliferation of possibilities, and reducing to some natural free field description takes some time. It is well worth it, despite the trouble – we will, in the end arrive at what amounts to linearized general relativity, a subject we shall return to later on (gravitational radiation is best described in terms of this model).

It is a long-ish road, the first-order form which was easy to get in E&M is not so obvious here, and this is again because of the free-wheeling index freedom provided by higher rank. In a sense, this is the last classical case we need to consider, and the fact that in the end, we get GR uniquely is a pleasant surprise.

We start with the general form of the action for a free, symmetric, massless second rank tensor field and whittle it down to a minimal description. Then, after a variety of variable changes, we finally arrive at a somewhat disguised first-order form and verify that the field equations are correct.

25.1 General, Symmetric Free Fields

We want to construct a field theory appropriate to a second rank tensor field $\tilde{h}_{\mu\nu}$ in four dimensions. For our Lagrangian, we can take any invariant, quadratic combinations of $\tilde{h}_{\mu\nu}$, $\tilde{h}_{\mu\nu,\gamma}$, $g_{\mu\nu} = \eta_{\mu\nu}$ (we are explicitly working with a Minkowski background) and $\epsilon^{\mu\nu\alpha\beta}$. As an additional twist, we will eventually specialize to symmetric tensor fields here, but *a priori*, we will assume nothing about the symmetries of $\tilde{h}_{\mu\nu}$.

As with the scalar and vector cases, the quadratic-in-the-field terms, like for example: $\tilde{h}^{\mu\nu}\tilde{h}_{\mu\nu}$ will correspond to adding "mass" to the field, and we are interested in the massless free field form (motivated by E&M and its massless A^μ). So we are left tabulating the quadratic derivative terms that could enter in the most general Lagrangian – to make life easier, define $\tilde{h}_{\mu\nu\alpha} \equiv \tilde{h}_{\mu\nu,\alpha}$, then we have:

$$\begin{aligned}
I &= \begin{cases} I_1 = \tilde{h}_{\mu\nu\alpha}\,\tilde{h}^{\mu\nu\alpha} & I_2 = \tilde{h}_{\mu\nu\alpha}\,h^{\alpha\mu\nu} \quad I_3 = \tilde{h}_{\mu\nu\alpha}\,\tilde{h}^{\nu\alpha\mu} \\ I_4 = \tilde{h}_{\mu\nu\alpha}\,\tilde{h}^{\nu\mu\alpha} & I_5 = \tilde{h}_{\mu\nu\alpha}\,\tilde{h}^{\alpha\nu\mu} \quad I_6 = \tilde{h}_{\mu\nu\alpha}\,\tilde{h}^{\mu\alpha\nu} \end{cases} \\
J &= \begin{cases} J_1 = \tilde{h}_{\mu\nu}{}^{\mu}\,\tilde{h}^{\nu\gamma}{}_{\gamma} & J_2 = \tilde{h}_{\mu\nu}{}^{\nu}\,\tilde{h}^{\mu\gamma}{}_{\gamma} \\ J_3 = \tilde{h}_{\mu\nu}{}^{\mu}\,\tilde{h}^{\gamma\nu}{}_{\gamma} & J_4 = \tilde{h}_{\mu\nu}{}^{\nu}\,\tilde{h}^{\gamma\mu}{}_{\gamma} \end{cases} \\
K &= \Big\{\; K_1 = \tilde{h}_{\mu\nu}{}^{\mu}\,\tilde{h}^{\gamma}{}_{\gamma}{}^{\nu} \quad K_2 = \tilde{h}_{\mu\nu}{}^{\nu}\,\tilde{h}^{\gamma}{}_{\gamma}{}^{\mu} \quad K_3 = \tilde{h}^{\mu}{}_{\mu\nu}\,\tilde{h}^{\gamma}{}_{\gamma}{}^{\nu} \quad .
\end{aligned} \tag{25.1}$$

I have grouped the above just to keep track of the various terms, the set I are three-three contractions (three indices from one $\tilde{h}$ contracted with three indices from another), J and K both contain combinations with one open index on each $\tilde{h}$ contracted with one open index from another – there are two different ways to close a pair, J and K represent combinations with both options.

The Lagrangian is a general combination of the above, so we begin with

$$\mathcal{L} = \sum_{\ell=1}^{6} \alpha_\ell\, I_\ell + \sum_{\ell=1}^{4} \beta_\ell\, J_\ell + \sum_{\ell=1}^{3} \gamma_\ell\, K_\ell. \tag{25.2}$$

For variation, it is best to write out the $\mathcal{L}$ in a total factored form – that way what we mean by $\frac{\partial \mathcal{L}}{\partial \tilde{h}_{\mu\nu,\alpha}}$ versus $\frac{\partial \mathcal{L}}{\tilde{h}^{\gamma}{}_{\gamma\mu}}$ will be clear – this is not strictly speaking necessary, but allows us to easily (if tediously) vary unambiguously

$$\begin{aligned}
\mathcal{L} = \tilde{h}_{\mu\nu,\alpha}\,\tilde{h}_{\rho\sigma,\gamma} \Big[& \alpha_1\, g^{\mu\rho}\, g^{\nu\sigma}\, g^{\alpha\gamma} + \alpha_2\, g^{\mu\sigma}\, g^{\nu\gamma}\, g^{\alpha\rho} + \alpha_3\, g^{\mu\gamma}\, g^{\nu\rho}\, g^{\alpha\sigma} \\
& + \alpha_4\, g^{\mu\sigma}\, g^{\nu\rho}\, g^{\alpha\gamma} + \alpha_5\, g^{\mu\gamma}\, g^{\nu\sigma}\, g^{\alpha\rho} + \alpha_6\, g^{\mu\rho}\, g^{\nu\gamma}\, g^{\alpha\sigma} \\
& + \beta_1\, g^{\mu\alpha}\, g^{\nu\rho}\, g^{\sigma\gamma} + \beta_2\, g^{\mu\rho}\, g^{\nu\alpha}\, g^{\sigma\gamma} + \beta_3\, g^{\mu\alpha}\, g^{\nu\sigma}\, g^{\rho\gamma} + \beta_4\, g^{\nu\alpha}\, g^{\mu\sigma}\, g^{\rho\gamma} \\
& + \gamma_1\, g^{\mu\alpha}\, g^{\nu\gamma}\, g^{\rho\sigma} + \gamma_2\, g^{\nu\alpha}\, g^{\mu\gamma}\, g^{\rho\sigma} + \gamma_3\, g^{\mu\nu}\, g^{\alpha\gamma}\, g^{\rho\sigma} \Big].
\end{aligned} \tag{25.3}$$

We vary according to the usual Euler-Lagrange prescription for fields, since there is no $\tilde{h}_{\mu\nu}$ dependence, the relevant portion for the field equations is just

$$\frac{\delta \mathcal{L}}{\delta \tilde{h}_{\beta\delta,\eta}} = 0 = G\,\tilde{h}_{\rho\sigma,\gamma}\left(\delta^{\beta}_{\mu}\,\delta^{\delta}_{\nu}\,\delta^{\eta}_{\alpha}\right) + G\,\tilde{h}_{\mu\nu,\alpha}\left(\delta^{\beta}_{\rho}\,\delta^{\delta}_{\sigma}\,\delta^{\eta}_{\gamma}\right) \tag{25.4}$$

with G defined to be the cubic metric terms in (25.3). This gives us a number of combinations, but notice that not all of the terms in the original Lagrangian are independent under variation, there are only seven combinations of coefficients that appear, compared with our initial set of thirteen coefficients. The field equations are

$$\begin{aligned}0 &= 2\,\alpha_1\,\tilde{h}^{\beta\delta\eta}{}_{,\,\eta} + (\alpha_2+\alpha_3+\beta_1+\beta_4)\left(\tilde{h}^{\eta\beta\delta}{}_{,\,\eta} + \tilde{h}^{\delta\eta\beta}{}_{,\,\eta}\right) + 2\,\alpha_4\,\tilde{h}^{\delta\beta\eta}{}_{,\,\eta} \\ &+ 2\,(\alpha_5+\beta_3)\,\tilde{h}^{\eta\delta\beta}{}_{,\,\eta} + 2\,(\alpha_6+\beta_2)\,\tilde{h}^{\beta\eta\delta}{}_{,\,\eta} + (\gamma_1+\gamma_2)\,(\tilde{h}^{\eta}{}_{\eta,}{}^{\delta\beta} + g^{\beta\delta}\,\tilde{h}^{\alpha\eta}{}_{,\alpha\eta}) \\ &+ 2\,\gamma_3\,g^{\beta\delta}\,\tilde{h}^{\eta}{}_{\eta,}{}^{\gamma}{}_{\gamma}.\end{aligned} \tag{25.5}$$

Looking at the above, we can make our first simplification – let us require that the field equations be symmetric in $\beta \leftrightarrow \delta$, this amounts to looking at symmetric $\tilde{h}_{\mu\nu}$.

The γ terms (coming from the invariants K) are already symmetric in (β, δ) interchange by virtue of the metric, and we see that to symmetrize the I and J sectors (constants α and β), we need:

$$\alpha_1 = \alpha_2 \qquad (\alpha_2+\alpha_3+\beta_1+\beta_4) = 2\,(\alpha_5+\beta_3) = 2\,(\alpha_6+\beta_2). \tag{25.6}$$

Then the field equations become:

$$\begin{aligned}0 &= 2\,\alpha_1\,(\tilde{h}^{\beta\delta\eta}{}_{,\,\eta} + \tilde{h}^{\delta\beta\eta}{}_{,\,\eta}) + 2\,(\alpha_5+\beta_3)\,(\tilde{h}^{\eta\beta\delta}{}_{,\,\eta} + \tilde{h}^{\delta\eta\beta}{}_{,\,\eta} + \tilde{h}^{\eta\delta\beta}{}_{,\,\eta} + \tilde{h}^{\beta\eta\delta}{}_{,\,\eta}) \\ &+ (\gamma_1+\gamma_2)\,(\tilde{h}^{\eta}{}_{\eta,}{}^{\delta\beta} + g^{\beta\delta}\,\tilde{h}^{\alpha\eta}{}_{,\alpha\eta}) + 2\,\gamma_3\,g^{\beta\delta}\,\tilde{h}^{\eta}{}_{\eta,}{}^{\gamma}{}_{\gamma}.\end{aligned} \tag{25.7}$$

The utility is clear – second derivatives commute, and we are left with symmetric combinations of $\tilde{h}_{\mu\nu}$. Define the symmetric form $h_{\mu\nu} \equiv \tilde{h}_{\mu\nu} + \tilde{h}_{\nu\mu}$, then we can write (noting that $h^{\gamma}{}_{\gamma} = 2\,\tilde{h}^{\gamma}{}_{\gamma}$)

$$0 = 2\,\alpha_1\,h^{\beta\delta\eta}{}_{,\,\eta} + 2\,(\alpha_5+\beta_3)\,(h^{\beta\eta\delta}{}_{,\,\eta} + h^{\delta\eta\beta}{}_{,\,\eta}) + 2\,(\gamma_1+\gamma_2)\,(h^{\eta}{}_{\eta,}{}^{\delta\beta} + g^{\beta\delta}\,h^{\alpha\eta}{}_{,\alpha\eta}) + 2\,\gamma_3\,g^{\beta\delta}\,h^{\eta}{}_{\eta,}{}^{\gamma}{}_{\gamma}. \tag{25.8}$$

So now we have a free, massless second rank field equation, and we want to "project out" the portion that is composed of pure first-rank tensor degenerate combinations. This is the same procedure as for our vector E&M case, where we wanted to make sure there was no $\phi_{,\mu}$ information in our field equations – here, the natural object is $A_{\mu,\nu} + A_{\nu,\mu}$, a symmetric second rank tensor built out of vectors. We will take $h_{\mu\nu} = A_{\mu,\nu} + A_{\nu,\mu}$ and remove any reference to $A_{\mu,\nu}$ in the field equations. Putting this into (25.8), we end up with the following constraints

$$\begin{aligned} 0 &= A^{\beta\delta\eta}{}_{,\,\eta}\,(2\,\alpha_1 + 2\,(\alpha_5+\beta_3)) \\ 0 &= A^{\eta\beta\delta}{}_{,\,\eta}\,(4\,(\alpha_5+\beta_3) + 4\,(\gamma_1+\gamma_2)) \\ 0 &= g^{\beta\delta}\,A^{\alpha\eta}{}_{,\alpha\eta}\,(4\,(\gamma_1+\gamma_2) + 4\,\gamma_3)\,, \end{aligned} \tag{25.9}$$

from which we learn that

$$\gamma_1+\gamma_2 = -\gamma_3 \qquad (\alpha_5+\beta_3) = \gamma_3 \qquad \alpha_1 = -\gamma_3 \tag{25.10}$$

and when the dust has settled, the field equations depend on an overall factor of γ_3 and nothing else:

$$\boxed{0 = \gamma_3\left(-h^{\beta\delta\eta}{}_{,\,\eta} + (h^{\beta\eta\delta}{}_{,\,\eta} + h^{\delta\eta\beta}{}_{,\,\eta}) - (h^{\eta}{}_{\eta,}{}^{\delta\beta} + g^{\beta\delta}\,h^{\alpha\eta}{}_{,\alpha\eta}) + g^{\beta\delta}\,h^{\eta}{}_{\eta,}{}^{\gamma}{}_{\gamma}\right)}\,. \tag{25.11}$$

This is the final form for a generic free massless second rank field. The constraints do not completely fix the action, there is still freedom in the choice of some coefficients (that do not appear in the field equations, effectively). We will fix these in a moment, but there is one last simplification we can introduce here.

25.1.1 Trace-Reversed Form

When terms like $h^{\eta}{}_{\eta,}{}^{\delta\beta}$ arise, it is sometimes useful to consider the trace-reversed form of the tensor field. This is just a notational shift, but one which can simplify both the field equations and the Lagrangian.

Let $h_{\mu\nu} = -H_{\mu\nu} + \frac{1}{2}\,g_{\mu\nu}\,H^{\alpha}{}_{\alpha}$, which can be inverted (take the trace) to define $H_{\mu\nu}$:

$$H_{\mu\nu} = -h_{\mu\nu} + \frac{1}{2}\,g_{\mu\nu}\,h^{\alpha}{}_{\alpha}, \tag{25.12}$$

using $g^{\alpha}{}_{\alpha} = D$, the dimension, which is four in this case.

Then the field equations become:

$$H^{\beta\delta\eta}{}_{,\,\eta} - H^{\beta\eta\delta}{}_{,\,\eta} - H^{\delta\eta\beta}{}_{,\,\eta} + g^{\beta\delta}\,H^{\alpha\eta}{}_{,\alpha\eta} = 0. \tag{25.13}$$

We can simplify the above – the scalar $H^{\alpha\eta}{}_{,\alpha\eta}$ can be solved in favor of the D'Alembertian of the trace. This is a common trick for re-organizing field equations. We have ten equations (β and δ are independent, but the field equations themselves are symmetric). If we take the $\beta-\delta$ trace of the above, we find

$$H^{\alpha\eta}{}_{,\alpha\eta} = -\frac{1}{2}\,H^{\sigma}{}_{\sigma,}{}^{\eta}{}_{\eta} \tag{25.14}$$

so we can write

$$H^{\beta\delta\eta}{}_{,\,\eta} - H^{\beta\eta\delta}{}_{,\,\eta} - H^{\delta\eta\beta}{}_{,\,\eta} - \frac{1}{2}\,g^{\beta\delta}\,H_{,\alpha}{}^{\alpha} = 0 \tag{25.15}$$

with $H \equiv H^{\alpha}_{\ \alpha}$. This will be our target from the Lagrangian.

25.1.2 First-Order Form

The general Lagrangian (25.3) written using the constraints (25.6) and (25.9) with the rest of the coefficient freedom fixed so as to give a symmetric spray of cubic metric (G from (25.4)) gives the simple form, in terms of $H^{\mu\nu}$:

$$\bar{\mathcal{L}} = 2\,\gamma_3\left(H^{\gamma}{}_{\gamma,\alpha}\,H^{\sigma}{}_{\sigma,}{}^{\alpha} + H_{\mu\nu,\alpha}\left(H^{\mu\alpha\nu}{}_{,} + H^{\nu\alpha\mu}{}_{,} - H^{\mu\nu\alpha}{}_{,}\right)\right). \tag{25.16}$$

From here, we can again write products of $H_{\mu\nu,\alpha}$ and symmetrize the metric products – this is just a matter of taste, and makes the canonical momenta a little easier to express. Explicitly

$$\begin{aligned}\bar{\mathcal{L}} = 2\,\gamma_3\,H_{\mu\nu,\alpha}\,H_{\rho\sigma,\gamma}\Bigg[&\frac{1}{2}\,g^{\mu\nu}\,g^{\rho\sigma}\,g^{\gamma\alpha} + \frac{1}{2}\,g^{\rho\mu}\,g^{\sigma\alpha}\,g^{\gamma\nu} + \frac{1}{2}\,g^{\sigma\nu}\,g^{\rho\alpha}\,g^{\gamma\nu} + \frac{1}{2}\,g^{\rho\nu}\,g^{\sigma\alpha}\,g^{\gamma\mu}\\ &+\frac{1}{2}\,g^{\sigma\nu}\,g^{\rho\alpha}\,g^{\gamma\mu} - \frac{1}{2}\,g^{\rho\mu}\,g^{\sigma\nu}\,g^{\gamma\alpha} - \frac{1}{2}\,g^{\sigma\mu}\,g^{\rho\nu}\,g^{\gamma\alpha}\Bigg].\end{aligned} \tag{25.17}$$

The procedure, as always, is to take $\pi^{\beta\delta\eta} = \frac{\partial\bar{\mathcal{L}}}{\partial H_{\beta\delta,\eta}}$, then the Legendre transform can be used to give us $\bar{\mathcal{H}}$ which will allow us to properly define the first-order form (where $\pi^{\mu\nu\alpha}$ and $H_{\mu\nu}$ are the independent variables). Using the definition, we find (written in covariant form for ease)

$$\pi_{\mu\nu\alpha} = 4\,\gamma_3\left[\frac{1}{2}\,H^{\gamma}{}_{\gamma,\alpha}\,H_{\mu\nu} + H_{\mu\alpha,\nu} + H_{\nu\alpha,\mu} - H_{\mu\nu,\alpha}\right]. \tag{25.18}$$

Our use of a symmetrized product of metrics allows us to see immediately that

$$\pi^{\mu\nu\alpha}\,H_{\mu\nu,\alpha} = 2\bar{\mathcal{L}} \tag{25.19}$$

numerically (which is to say, if we use (25.18)). So our Legendre transform, obtained by calculating

$$\bar{\mathcal{H}} = \pi_{\mu\nu\alpha}\,H^{\mu\nu\alpha}{}_{,} - \bar{\mathcal{L}} \tag{25.20}$$

is itself equal to $\bar{\mathcal{L}}$. Of course, $\bar{\mathcal{H}}$ is a function only of $\pi_{\mu\nu\alpha}$, so there is a moral difference.

First order form will be achieved by taking the resulting $\bar{\mathcal{H}}$ and Legendre transforming back to $\bar{\mathcal{L}}$, from which the variation, independently, of $H_{\mu\nu,\alpha}$ and $\pi^{\mu\nu\alpha}$ will proceed. If we first invert the relation (25.18) to find $H_{\mu\nu,\alpha}$ as a function of $\pi^{\mu\nu\alpha}$ (define $\gamma \equiv 4\,\gamma_3$ for the following):

$$H_{\mu\nu,\alpha} = \frac{1}{2\,\gamma}\left(\pi_{\mu\alpha\nu} + \pi_{\nu\alpha\mu} - \frac{1}{3}\left(g_{\mu\alpha}\,\pi^{\gamma}{}_{\nu\gamma} + g_{\alpha\nu}\,\pi^{\gamma}{}_{\mu\gamma}\right)\right) \tag{25.21}$$

then we can write $\bar{\mathcal{H}} = \bar{\mathcal{L}}|_{H_{\mu\nu,\alpha}(\pi_{\mu\nu\alpha})}$ as

$$\bar{\mathcal{H}} = \frac{1}{2\,\gamma}\left[\frac{1}{2}\,\pi^{\mu\nu\alpha}\left(\pi_{\mu\alpha\nu} + \pi_{\nu\alpha\mu}\right) - \frac{1}{3}\,\pi^{\sigma}{}_{\alpha\sigma}\,\pi^{\gamma\alpha}{}_{\gamma}\right]. \tag{25.22}$$

So, finally, we have the Lagrange density in true first-order form: take $\pi^{\mu\nu\alpha}\,H_{\mu\nu,\alpha} - \bar{\mathcal{H}}$ at face value, this is precisely $\bar{\mathcal{L}}$:

$$\bar{\mathcal{L}} = \pi^{\mu\nu\alpha}\,H_{\mu\nu,\alpha} - \frac{1}{2\,\gamma}\left(\frac{1}{2}\,\pi^{\mu\nu\alpha}\left(\pi_{\mu\alpha\nu} + \pi_{\nu\alpha\mu}\right) - \frac{1}{3}\,\pi^{\sigma}{}_{\alpha\sigma}\,\pi^{\gamma\alpha}{}_{\gamma}\right). \tag{25.23}$$

25.1.3 Checking the Field Equations

The advantage of (25.23) is that the variation w.r.t. $H^{\mu\nu}{}_{,}{}^{\alpha}$ is trivial – it returns, just as for E&M, the divergence-less character of $\pi^{\mu\nu\alpha}$, the canonical momentum (we had $F^{\mu\nu}{}_{,\nu} = 0$ for electrodynamics):

$$\frac{\delta\bar{\mathcal{L}}}{\delta H_{\mu\nu,\alpha}} = \pi^{\mu\nu\alpha}{}_{,\alpha} = 0. \tag{25.24}$$

Then all we need is for the variation w.r.t. $\pi^{\mu\nu\alpha}$ to give the correct definition of $\pi^{\mu\nu\alpha}$ in terms of $H_{\mu\nu,\alpha}$. From (25.18), we know that $\pi^{\mu\nu\alpha} = \pi^{\nu\mu\alpha}$, the momentum inherits the symmetry of $H_{\mu\nu}$. We have to be a little careful in our variation, then. Forgetting about the symmetry for a moment, we have

$$\begin{aligned}\delta\bar{\mathcal{L}} &= H_{\mu\nu,\alpha}\,\delta\pi^{\mu\nu\alpha} - \frac{1}{2\,\gamma}\left[\frac{1}{2}\left(\pi_{\mu\alpha\nu} + \pi_{\nu\alpha\mu} + \pi_{\mu\alpha\nu} + \pi_{\alpha\mu\nu}\right)\delta\pi^{\mu\nu\alpha} - \frac{2}{3}\,\pi^{\sigma}{}_{\alpha\sigma}\,\delta\pi^{\gamma\alpha}{}_{\gamma}\right] \\ &= \left[H_{\mu\nu,\alpha} - \frac{1}{2\,\gamma}\left(\pi_{\mu\alpha\nu} + \frac{1}{2}\left(\pi_{\nu\alpha\mu} + \pi_{\alpha\mu\nu}\right) - \frac{2}{3}\,\pi^{\gamma}{}_{\nu\gamma}\,g_{\mu\alpha}\right)\right]\delta\pi^{\mu\nu\alpha},\end{aligned} \tag{25.25}$$

and the symmetry comes in observing that it is the symmetric portion (in $\mu \leftrightarrow \nu$) of the above term in brackets that must vanish – we can say nothing about the antisymmetric part, which is killed automatically. Using the symmetry of the field $H_{\mu\nu}$ itself, we have, finally:

$$H_{\mu\nu,\alpha} - \frac{1}{2\,\gamma}\left(\pi_{\mu\alpha\nu} + \pi_{\nu\alpha\mu} - \frac{1}{3}\left(g_{\mu\alpha}\,\pi^{\gamma}{}_{\nu\gamma} + g_{\nu\alpha}\,\pi^{\gamma}{}_{\mu\gamma}\right)\right) = 0, \tag{25.26}$$

and comparing with (25.21), we see that this field equation enforces the relation between $H_{\mu\nu,\alpha}$ and $\pi^{\mu\nu\alpha}$. This is no surprise, it is the whole point of the Hamiltonian (first order form) approach. If we uninvert, to get $\pi^{\mu\nu\alpha}\,(H_{\mu\nu,\alpha})$ (just (25.18)) then take the divergence $\pi^{\mu\nu\alpha}{}_{,\alpha} = 0$ as dictated by the other field equation, we will get the original trace-reversed field equation for $H_{\mu\nu}$ (25.15). Again, because $\frac{\partial \bar{\mathcal{L}}}{\partial H_{\mu\nu,\alpha}} \equiv \pi^{\mu\nu\alpha}$ defines the momenta, the field equation for $H_{\mu\nu}$ *must* follow from the divergence since the original field equations for $\bar{\mathcal{L}}$ in terms of $H_{\mu\nu,\alpha}$ (25.16) are precisely $\partial_\alpha\left(\frac{\partial \bar{\mathcal{L}}}{\partial H_{\mu\nu,\alpha}}\right) = 0$ owing to the lack of $H_{\mu\nu}$ (with no derivative) in the starting Lagrangian (25.2).

25.1.4 A Change of Momenta

Finally, we will rewrite the canonical momentum $\pi^{\mu\nu\alpha}$ in terms of a slightly modified triply-indexed object. This change is equivalent to the move from $h_{\mu\nu}$ to the trace-reversed $H_{\mu\nu}$. Indeed, what we are effectively doing is using $H_{\mu\nu}$ as the field, but the momentum appropriate to the original $h_{\mu\nu}$. That's a technical point, and I don't want to belabor it – the utility will become clear when we connect all of this to the natural geometric objects that we sort of (and only "sort of" at this stage) expect to see in the full Lagrangian of general relativity. From our current point of view, I am just going to dispense with $\pi^{\mu\nu\alpha}$ in favor of a new object which can be varied independently. Define $\Gamma_{\alpha\mu\nu}$ by

$$\pi_{\mu\nu\alpha} = \gamma\,\left[-2\,\Gamma_{\alpha\mu\nu} + g_{\mu\alpha}\,\Gamma^{\gamma}{}_{\nu\gamma} + g_{\nu\alpha}\,\Gamma^{\gamma}{}_{\mu\gamma}\right], \tag{25.27}$$

then inputting this definition in (25.23) gives

$$\bar{\mathcal{L}} = 2\,\gamma\left(\left[-\Gamma_{\alpha,\mu\nu} + g_{\mu\alpha}\,\Gamma^{\gamma}{}_{\nu\gamma}\right]\,H^{\mu\nu}{}_{,}{}^{\alpha} + \left[\Gamma^{\alpha\sigma}{}_{\sigma}\,\Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma_{\mu\nu\alpha}\,\Gamma^{\alpha\mu\nu}\right]\right). \tag{25.28}$$

For what follows in the next few sections, we will take $H^{\mu\nu}$ to be the fundamental field, which leads to a modified momentum naturally described in terms of $\Gamma^{\alpha}{}_{\mu\nu}$ – using this, and the fact that for $\bar{\mathcal{L}}$ going into the action, we can flip derivatives while picking up minus signs (integration by parts), we have the final (final) form:

$$\boxed{\bar{\mathcal{L}} = H^{\mu\nu} \left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} \right) + g^{\mu\nu} \left(\Gamma^{\alpha}{}_{\mu\nu} \, \Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu} \, \Gamma^{\beta}{}_{\alpha\nu} \right).} \tag{25.29}$$

The naming here, $\Gamma^{\alpha}{}_{\mu\nu}$ is no accident – it was presaged by (25.18) which has a familiar combination of derivatives of $H_{\mu\nu,\alpha}$ in it. In the final form, we will see that $g_{\mu\nu} + h_{\mu\nu}$ can be interpreted as the metric and $\Gamma^{\alpha}{}_{\mu\nu}$ is a connection – in fact, from the field equations (as is already evident in (25.21)), it is the unique connection associated with a metric space, and this will give us the usual geometric interpretation of general relativity.