Second Rank Consistency

Lecture 26

Physics 411
Classical Mechanics II

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We take our free second rank field theory and introduce some external coupling (to another field theory, say). This results, as it did for E&M, in an inconsistency – basically the loss of conservation laws, which indicate that the combined theory for $H^{\mu\nu}$ and some other field ϕ (for example) is inconsistent with coordinate invariance. At the crux of the problem is the role of $H^{\mu\nu}$ as its own source. We have not encountered this in earlier field theories, since we have seen no fields that carry source "charges". In demanding self-consistency when coupled to "matter" (our generic term for any other field), the usual geometric interpretation of the field $H^{\mu\nu}$ becomes clear[1].

26.1 Modified Action Variation

When we were done, the final action form read:

$$\bar{\mathcal{L}} = H^{\mu\nu} \left(\Gamma^{\alpha}_{\ \mu\nu,\alpha} - \Gamma^{\alpha}_{\ \mu\alpha,\nu} \right) + g^{\mu\nu} \left(\Gamma^{\alpha}_{\ \mu\nu} \Gamma^{\gamma}_{\ \alpha\gamma} - \Gamma^{\alpha}_{\ \beta\mu} \Gamma^{\beta}_{\ \alpha\nu} \right), \tag{26.1}$$

which will give two sets of "first-order" equations, corresponding to variation w.r.t. $H^{\mu\nu}$ and $\Gamma^{\alpha}_{\ \beta\gamma}$. We can combine the two to give one second order equation, the same one we've always had. But it is interesting to look at the first-order results alone. Varying w.r.t. $H^{\mu\nu}$, we find that

$$\boxed{\Gamma^{\alpha}_{\ \mu\nu,\alpha} - \Gamma^{\alpha}_{\ \mu\alpha,\nu} = 0,} \tag{26.2}$$

[1]This is all nicely laid out in its original form: S. Deser, "Self-Interaction and Gauge Invariance," *Gen. Rel. Grav.* 1, 1970, gr-qc/0411023.

and varying w.r.t. $\Gamma^\alpha{}_{\beta\gamma}$ effectively gives us the relation between derivatives of $H_{\mu\nu}$ and the momenta – we have

$$0 = -H^{\rho\delta}{}_{,\sigma} + \frac{1}{2}\left(H^{\rho\nu}{}_{,\nu}\,\delta^\delta_\sigma + H^{\delta\nu}{}_{,\nu}\,\delta^\rho_\sigma\right) + g^{\rho\delta}\,\Gamma^\gamma{}_{\sigma\gamma} + \frac{1}{2}\,g^{\mu\nu}\left(\Gamma^\rho{}_{\mu\nu}\delta^\delta_\sigma + \Gamma^\delta{}_{\mu\nu}\delta^\rho_\sigma\right) - \left(g^{\delta\nu}\,\Gamma^\rho_{\sigma\nu} + g^{\rho\nu}\,\Gamma^\delta{}_{\sigma\nu}\right). \tag{26.3}$$

If we input the definition $H_{\mu\nu} = -h_{\mu\nu} + \frac{1}{2}\,g_{\mu\nu}h$, then the above boils down to:

$$\boxed{\Gamma_{\alpha\mu\nu} = \frac{1}{2}\,(h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}).} \tag{26.4}$$

At this point, I want to make sure that the importance of this *as a field equation* does not escape us. We are being told that our theory, a fairly general free second-rank tensor field has, built in somehow, the idea that the field itself defines what amounts to the usual metric connection we get out of curvilinear coordinates (for example). Indeed the above (with h replaced by g) is valid even when the underlying space is not flat. Somehow, making a relativistic symmetric second rank field theory is *already* tied to some geometrical notions.

Let's now return to the original field equation (26.2) – you might wonder if this relation is also of geometrical interest – it is, but we'll return to that point in a bit. For now, I want to note the similarity with the electromagnetic case, where the natural analogue of the above was $\partial_\mu F^{\mu\nu} = 0$. Here we have a similar situation, derivatives of the "momenta" are zero. Recall from E&M, that we had in addition to the field equation, an *a priori* relation based on the antisymmetry of $F^{\mu\nu}$, namely; $\partial_\mu\partial_\nu F^{\mu\nu} = 0$. This invariance is what leads, pretty directly to charge conservation once charge is introduced as a source – when we coupled matter to E&M via a term like $A_\mu\,J^\mu$ in the action, the field equations read

$$\partial_\mu\,F^{\nu\mu} \sim J^\nu \longrightarrow \partial_\mu\partial_\nu\,F^{\nu\mu} \sim \partial_\nu J^\nu = 0. \tag{26.5}$$

The object appearing in (26.2) is the (linearized) Ricci tensor, defined as:

$$\tilde{R}_{\mu\nu} \equiv \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu}. \tag{26.6}$$

With our relation between Γ and h in place, we can write the Ricci tensor as a function of the field $h_{\mu\nu}$:

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\,g^{\alpha\rho}\,(h_{\rho\nu,\mu\alpha} - h_{\mu\nu,\rho\alpha} - h_{\rho\alpha,\mu\nu} + h_{\mu\alpha,\rho\nu})\,. \tag{26.7}$$

Clearly, once the field equation (26.2) is satisfied, this is zero, that's just like saying $\partial_\mu F^{\mu\nu} = 0$ in electrodynamics. But the linearized Ricci tensor has a symmetry relation for its derivatives much like $\partial_\mu \partial_\nu F^{\mu\nu} = 0$. Let's see what the divergence of this tensor is:

$$\partial^\mu \tilde{R}_{\mu\nu} = \frac{1}{2} \left(h^{\mu\alpha}{}_{,\mu\alpha\nu} - h_{,}{}^{\mu}{}_{\mu\nu} \right) \tag{26.8}$$

with $h \equiv h^\alpha{}_\alpha$ as usual. This doesn't appear to be zero, but consider the linearized Ricci scalar obtained by contracting the tensor:

$$\tilde{R} = h^{\mu\alpha}{}_{,\mu\alpha} - h_{,}{}^{\mu}{}_{\mu}, \tag{26.9}$$

we see that by taking the ν derivative of this we could kill the terms in the divergence of $\tilde{R}_{\mu\nu}$ – this suggests that we form the combination:

$$\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R}, \tag{26.10}$$

which is the (linearized) "Einstein Tensor" – *this* tensor has the property that $\partial^\mu \tilde{G}_{\mu\nu} = 0$ automatically. The relation is known as the "Bianchi identity" (linearized form) and is an intrinsic property of the definitions of the Ricci and Einstein tensors having nothing to do with the dynamics of $h_{\mu\nu}$.

How will we introduce a coupling of $h_{\mu\nu}$ to sources? Following E&M, where we took a charge-current density J^μ and coupled it to the vector A_μ via $J^\mu A_\mu$, we evidently need a second rank tensor to contract with $h_{\mu\nu}$ in order to get a source for the right-hand side of (26.2). What symmetric second rank tensors that describe sources do we know? The stress-energy tensor. The source for this field theory must be the full $T^{\mu\nu}$ of, for example, fluids or particles (or any other field). This is our first indication that whatever theory we are building has some sort of universal coupling. Remember that Newtonian gravity, as a scalar field, couples to ρ, the mass density (the T^{00} component of the stress tensor). E&M as a vector field couples to charge-current (a vector), the $T^{0\mu}$ component of a charged particle stress tensor. Evidently, our second rank tensor field theory couples to the density ρ, as well as any "mass-current" (through T^{0i}), and indeed the rest of the stress tensor as well. In addition, we are freed from discussing "just" mass – after all, E&M as a field theory has a stress tensor, the electric and magnetic fields carry energy and our $h_{\mu\nu}$ theory will also be sourced by that.

This is all very good as a theory of universal gravitation – we take $E = mc^2$ and invert it, then energy as a "mass source" is natural and we expect a theory of gravity to couple to all forms of energy, not just the traditional "stuff". But now we can begin to see the issue – as a field theory, $h_{\mu\nu}$ *itself* has a stress-energy tensor, we know how to form it – but then why wouldn't the stress tensor of $h_{\mu\nu}$ *also source* $h_{\mu\nu}$? The answer, from the current linearized point of view is that it does, and it is precisely this self-coupling that we need to introduce. That's the moral of the story, and we will now look at the mathematical procedure for re-coupling. As a final note, think about what all of this would mean for E&M – the big difference between E&M and GR is that the fields in GR carry "charge" (in the sense of energy) that sources the fields, while in E&M, the electromagnetic field does *not* carry any charge, and so does not generate itself (electromagnetic waves can propagate through space, of course, that's not a source of fields).

How is all of this going to manifest itself in the field equations? Think of the simplest possible field we could couple $H_{\mu\nu}$ to – a scalar field. If we took our $\bar{\mathcal{L}}_H$ for $H_{\mu\nu}$ and introduced a term like $H_{\mu\nu}T^{\mu\nu}$ for the stress-tensor of a scalar field, we would get the above (26.3) field equation, but the (26.2) one would become: $\tilde{R}_{\mu\nu} \sim T_{\mu\nu}$. Fine, but do we now know that $\partial^\mu T_{\mu\nu} = 0$? The Lagrangian for a free massless scalar field is $\bar{\mathcal{L}}_\phi = \phi_{,\mu}\,\phi_{,\nu}\,g^{\mu\nu}$, and the associated stress tensor is

$$T^{\mu\nu}_\phi = \phi_{,}{}^{\mu}\phi_{,}^{\nu} - \frac{1}{2}\,g^{\mu\nu}\left(\phi_{,\alpha}\phi_{,}{}^{\alpha}\right), \tag{26.11}$$

so its divergence is

$$\partial_\mu\,T^{\mu\nu}_\phi = \phi_{,\mu}^{\ \ \mu}\,\phi_{,\nu} = \Box^2\phi\,\partial_\nu\,\phi \tag{26.12}$$

which vanishes *only if* the field ϕ satisfies $\phi_{,\mu}{}^{\mu} = 0$, the field equation for a free scalar. So in the absence of coupling, we have a conserved stress tensor, but the $H_{\mu\nu}\,T^{\mu\nu}_\phi$ term doesn't just modify the equation (26.2) under variation of $H^{\mu\nu}$, it also appears in the scalar field theory. This is all the same story we encountered when coupling complex scalar fields to an electromagnetic field – there we started with a current that was conserved in the free theory, and had to modify the current by changing our definition of derivative in order to make the coupled theory self-consistent.

In the tensor case, we will end up with a Lagrangian like:

$$\begin{aligned}\bar{\mathcal{L}}_{H,\phi} =& \Big(H^{\mu\nu}\left(\Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu}\right) + g^{\mu\nu}\left(\Gamma^\alpha{}_{\mu\nu}\,\Gamma^\gamma{}_{\alpha\gamma} - \Gamma^\alpha{}_{\beta\mu}\,\Gamma^\beta{}_{\alpha\nu}\right)\Big) \\ &+ H^{\mu\nu}\left(\phi_{,\mu}\,\phi_{,\nu} - \frac{1}{2}\,g_{\mu\nu}\left(\phi_{,\alpha}\,\phi_{,}{}^{\alpha}\right)\right) + \left(\phi_{,\mu}\,\phi_{,\nu}\,g^{\mu\nu}\right)\end{aligned} \tag{26.13}$$

and the field equations for ϕ will now look like

$$2\Box^2\phi + 2\,\partial_\nu\left(H^{\mu\nu}\,\phi_{,\mu}\right) - \partial_\alpha\left(H\phi_{,\alpha}\right) = 0 \qquad (26.14)$$

which upon insertion in (26.12) will not give zero. So we have an inconsistency – on the one hand, $\tilde{R}_{\mu\nu} \sim T_{\mu\nu}$ demands that $\partial^\mu T_{\mu\nu} = 0$, but the modified field equations for ϕ will not allow this.

The salvation comes in the observation that $H^{\mu\nu}$ itself has a stress tensor, and this must be coupled to $H^{\mu\nu}$ for the matter-coupled theory to be consistent. With no external stress tensor, everything is fine, $\tilde{R}_{\mu\nu} = 0$ and so too does its divergence. So in the free field version, what we are about to do is unnecessary – it restores what would be a problem down the road.

26.2 Stress Tensor for $h_{\mu\nu}$

Let's go back to the stress tensor that would arise from a theory for $h_{\mu\nu}$. We have not made a scalar action for curvilinear coordinates, starting as we did from the assumption that $g^{\mu\nu} = \eta^{\mu\nu}$ for flat Cartesian coordinates – but in theory, we can fix this trivially. Notice also that this is the first action we have considered that *needs* fixing – the scalar theory used "normal" partials since $\phi_{,\mu} = \phi_{;\mu}$, and the vector theory used $A_{\mu,\nu} - A_{\nu,\mu}$ which kills the connection term automatically. But our starting action can be written as

$$\bar{\mathcal{L}} = \left(-H^{\mu\nu}{}_{,\alpha}\,\Gamma^\alpha{}_{\mu\nu} + H^{\mu\nu}{}_{,\nu}\,\Gamma^\alpha{}_{\mu\alpha}\right) + g^{\mu\nu}\left(\Gamma^\alpha{}_{\mu\nu}\,\Gamma^\gamma{}_{\alpha\gamma} - \Gamma^\alpha{}_{\beta\mu}\,\Gamma^\beta{}_{\alpha\nu}\right), \qquad (26.15)$$

and we can easily extend this form to a scalar by introducing semicolons for commas. In addition, we must interpret $H^{\mu\nu}$ as a tensor density, i.e. as a tensor multiplied by $\sqrt{-g}$ here, this means that $\mathcal{L} = \bar{\mathcal{L}}$ formally, and we do this only for simplicity.

Remember that the stress tensor is formed from coordinate invariance – that is, $\frac{\delta\bar{\mathcal{L}}}{\delta g^{\mu\nu}} \sim T_{\mu\nu}$ comes from the assumption that the action is unchanged by (infinitesimal) coordinate transformation. We want to now calculate the derivative of $\bar{\mathcal{L}}$ w.r.t. $g_{\mu\nu}$ (we will use the upper form for now), that will describe the stress tensor associated with the field (on flat space) $H^{\mu\nu}$. For notation, write the connection in our curvilinear coordinate system as $C^\alpha{}_{\beta\gamma}$, just to distinguish it from the momenta for $H^{\mu\nu}$. Then the variational derivative w.r.t. $g^{\mu\nu}$ will be

$$\frac{\delta\bar{\mathcal{L}}}{\delta g^{\mu\nu}} = \left(2\, H^{\sigma\nu}\,\Gamma^{\alpha}{}_{\mu\nu} - H^{\sigma\alpha}\,\Gamma^{\rho}{}_{\mu\rho} - H^{\gamma\sigma}\,\Gamma^{\rho}{}_{\gamma\rho}\,\delta^{\alpha}_{\mu}\right)\,\frac{\partial C^{\mu}{}_{\sigma\alpha}}{\partial g^{\mu\nu}} + \left(\Gamma^{\alpha}{}_{\mu\nu}\,\Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu}\,\Gamma^{\beta}{}_{\alpha\nu}\right). \tag{26.16}$$

We could calculate the derivative of the connection w.r.t. $g^{\mu\nu}$ and use it as part of the stress tensor, but as it turns out, we don't actually need it (the resulting contribution to the field equations does not itself generate an inconsistent stress tensor). In the end, only the $\Gamma\,\Gamma$ terms contribute. That means that if we want to "couple" our $H_{\mu\nu}$ field theory to itself (which we must do before coupling it to anything else), we need the action to be

$$\bar{\mathcal{L}} = H^{\mu\nu}\left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu}\right) + g^{\mu\nu}\left(\Gamma^{\alpha}{}_{\mu\nu}\,\Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu}\,\Gamma^{\beta}{}_{\alpha\nu}\right) + H^{\mu\nu}\left(\Gamma^{\alpha}{}_{\mu\nu}\,\Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu}\,\Gamma^{\beta}{}_{\alpha\nu}\right). \tag{26.17}$$

Notice that, by construction, the second and third terms differ only by the factor in front, so we can combine these in a form with overall $(g^{\mu\nu} + H^{\mu\nu})$. In addition, we can add a term that looks like $g^{\mu\nu}\left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu}\right)$ with impunity, since the derivatives can be flipped onto the underlying flat metric which will vanish. Overall, then, we find that the original theory for $H_{\mu\nu}$, when made self-consistent by coupling to its own stress tensor becomes:

$$\boxed{\bar{\mathcal{L}} = (g^{\mu\nu} + H^{\mu\nu})\left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\mu\nu}\,\Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu}\,\Gamma^{\beta}{}_{\alpha\nu}\right).} \tag{26.18}$$

If we vary w.r.t. $H^{\mu\nu}$, or even the additive combination $(g^{\mu\nu} + H^{\mu\nu})$, we will now obtain the relation:

$$\left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\mu\nu}\,\Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu}\,\Gamma^{\beta}{}_{\alpha\nu}\right) = 0, \tag{26.19}$$

which is the *full* Ricci tensor, $R_{\mu\nu} = 0$. Varying w.r.t. $\Gamma^{\alpha}{}_{\beta\gamma}$ will return (26.3) but with $h_{\mu\nu} \to h_{\mu\nu} + g_{\mu\nu}$. This second equation is important – it tells us that for our new and improved theory, the "momenta" defines $g_{\mu\nu} + h_{\mu\nu}$ as the metric on a manifold with a connection. The point is, our combination, as a feature of the field equation, can be viewed as a metric. Then the condition $R_{\mu\nu} = 0$ gives us information about the geometric properties of that space. In this way, the "field" of interest here will end up being the metric itself, and its field equation becomes a constraint on the geometry of the spaces defined by the metric.

The original goal here was to modify the free field (26.1) to accomodate coupling to external stress tensors. Our next job is to show that we have indeed accomplished this. Again, think of E&M, where the self-consistency of the theory coupled to scalar fields (for example) was restored by introducing a modified "covariant derivative". In addition, we were then able to drop the explicit coupling term $A_\mu J^\mu$, all the coupling was through the A_μ inside the covariant derivative.

Our new second rank theory is no different, we will introduce a "covariant derivative" involving the "field" (metric) and write all of our external actions in terms of this new derivative. In addition, the usual scalar volume element $\sqrt{-g}\, d\tau$ term appearing in most actions will ensure that everything depends on $g_{\mu\nu}$.