Final Form and Matter Coupling

Lecture 27

Physics 411
Classical Mechanics II

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We have seen that a self-consistent (in the presence of sources) second rank tensor field theory is necessarily nonlinear. In addition, the field equations themselves indicate the interpretation of the field $g_{\mu\nu}$ as the metric on a Riemannian space-time. In this sense, general relativity, as a second rank field theory, is highly constrained.

The goal now is to finish the story – introduce the proper coupling to matter, which follows precisely the same structure as coupling currents to E&M. The usual statement: "Matter is the source for a curved space-time" can then be given some meaning. What we will find is that when we combine any theory with GR, the metric is determined from Einstein's equation, and this metric provides the arena for the external theory.

So, we will have the relation between the sources and field, and after we have solved for some particular sources (i.e. none), we can turn our attention to test particle motion. Here, too, what we end up with is motion in a curved space-time.

27.1 Full Field Equations

We saw that the scalar Lagrangian could be written as

$$\bar{\mathcal{L}} = g^{\mu\nu} \underbrace{\left(\Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\mu\nu} \Gamma^{\gamma}{}_{\alpha\gamma} - \Gamma^{\alpha}{}_{\beta\mu} \Gamma^{\beta}{}_{\alpha\nu} \right)}_{\equiv R_{\mu\nu}}, \tag{27.1}$$

where now we have soaked the dynamical field $H^{\mu\nu}$ into $g^{\mu\nu}$ – the point is that $H^{\mu\nu}$ always appears in combination with $g^{\mu\nu}$ (the background metric) and plays the role of a metric, so we cannot actually distinguish between the two. From now on, then $g^{\mu\nu}$ is the whole story – our second rank, self-consistent field theory has, as its field, the metric of some pseudo-Riemannian space(-time). The action that this comes from is

$$\boxed{S = \int d\tau \, \sqrt{-g} \, g^{\mu\nu} \, R_{\mu\nu}(\Gamma)} \tag{27.2}$$

and we have been careful to put in the density $\sqrt{-g}$ to counteract $d\tau$. The integrand is clearly a scalar, so we will obtain a valid covariant theory.

Let's do the variation one last time, with the full $g^{\mu\nu}$ in place – we will take the first-order point of view, where as always, the field $g^{\mu\nu}$ and momenta $\Gamma^{\alpha}{}_{\beta\gamma}$ are varied separately. Varying w.r.t. $g^{\mu\nu}$ is easy[1] if we view $R_{\mu\nu}$ as a function of the momenta $\Gamma^{\alpha}{}_{\beta\gamma}$ and its derivatives, there is no metric dependence and

$$\begin{aligned} \delta S &= \sqrt{-g} \, R_{\alpha\beta} \, \delta g^{\alpha\beta} - \frac{1}{2} \, R_{\mu\nu} \, g^{\mu\nu} \, \frac{\partial g}{\partial g^{\alpha\beta}} \, \delta g^{\alpha\beta} \\ &= \sqrt{-g} \left(R_{\alpha\beta} - \frac{1}{2} \, g_{\alpha\beta} \, R \right) \, \delta g^{\alpha\beta}. \end{aligned} \tag{27.3}$$

This tells us that the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} \, g_{\mu\nu} \, R = 0$, which is precisely Einstein's equation in vacuum. We do not yet know that $g_{\mu\nu}$ is a metric – there is currently no relationship between the metric and the momenta, in terms of which $R_{\mu\nu}$ (and hence $G_{\mu\nu}$) is written. As usual, the relation is set by the $\Gamma^{\alpha}{}_{\beta\gamma}$ variation. This is a little more involved, but still relatively easy. If we introduce the metric density $\mathfrak{g}^{\mu\nu} \equiv \sqrt{-g} \, g^{\mu\nu}$, then variation gives

$$\delta S = \left(-\mathfrak{g}^{\mu\nu}{}_{,\alpha} + \mathfrak{g}^{\mu\gamma}{}_{,\gamma} \, \delta^{\nu}_{\alpha} + \Gamma^{\gamma}{}_{\alpha\gamma} \, \mathfrak{g}^{\mu\nu} - 2 \, \Gamma^{\mu}{}_{\alpha\rho} \, \mathfrak{g}^{\nu\rho} + \mathfrak{g}^{\rho\sigma} \, \Gamma^{\mu}{}_{\rho\sigma} \, \delta^{\nu}_{\alpha} \right) \, \delta\Gamma^{\alpha}{}_{\mu\nu}, \tag{27.4}$$

and we require that this be zero – there is a slight catch here – the connection (momenta) $\Gamma^{\alpha}{}_{\mu\nu}$ is symmetric in $\mu \leftrightarrow \nu$, so that only the symmetric part of the term in parenthesis needs to be zero for the variation to vanish. Our field equation becomes

$$-\mathfrak{g}^{\mu\nu}{}_{,\alpha} + \frac{1}{2} \left(\mathfrak{g}^{\mu\gamma}{}_{,\gamma} \, \delta^{\nu}_{\alpha} + \mathfrak{g}^{\nu\gamma}{}_{,\gamma} \, \delta^{\mu}_{\alpha} \right) + \Gamma^{\gamma}{}_{\alpha\gamma} \, \mathfrak{g}^{\mu\nu} - \Gamma^{\mu}{}_{\alpha\rho} \, \mathfrak{g}^{\nu\rho} - \Gamma^{\nu}{}_{\alpha\rho} \, \mathfrak{g}^{\mu\rho} + \frac{1}{2} \, \mathfrak{g}^{\rho\sigma} \left(\Gamma^{\mu}{}_{\rho\sigma} \, \delta^{\nu}_{\alpha} + \Gamma^{\nu}{}_{\rho\sigma} \, \delta^{\mu}_{\alpha} \right) = 0. \tag{27.5}$$

The long road is to introduce [2]

$$\mathfrak{g}^{\mu\nu}{}_{,\alpha} = \sqrt{-g}_{,\alpha} \, g^{\mu\nu} + \sqrt{-g} \, g^{\mu\nu}{}_{,\alpha} = \sqrt{-g} \left(\frac{1}{2} \, g^{\rho\sigma} \, g^{\mu\nu} \, g_{\rho\sigma,\alpha} + g^{\mu\nu}{}_{,\alpha} \right), \tag{27.6}$$

[1] note that $\frac{\partial g}{\partial g_{\mu\nu}} = g \, g^{\mu\nu}$, which directly implies that $\frac{\partial g}{\partial g^{\mu\nu}} = -g \, g_{\mu\nu}$.

[2] The coordinate derivative of the density g is $g_{,\alpha} = g \, g^{\rho\sigma} \, g_{\rho\sigma,\alpha}$.

then replace all partials with covariant derivatives, massage, and voila – we get

$$g_{\mu\nu;\alpha} = 0 \tag{27.7}$$

which yields the usual relationship between connection and metric.

This is no surprise, just a check of what we already knew. The logic is: Variation w.r.t. $g^{\mu\nu}$ gives the Ricci-flat condition, a statement about allowed geometry, while variation w.r.t. $\Gamma^{\alpha}_{\ \mu\nu}$ connects the field $g^{\mu\nu}$ to a metric space. In vacuum, it is easy to see that the vanishing of the Einstein tensor demands that $R_{\mu\nu} = 0$, so that here, we need not keep the full varied form: $R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0$, but this is useful for coupling.

The "first-order" formalism allowed us to treat this action in a Hamiltonian fashion, just as we did with E&M. In this GR setting, the action written in this way is known as "Palatini Form".

27.2 Matter

The vacuum Einstein equations can be solved by themselves, and that is a project we will take up later on, but we have not yet described the relation between sources and $g_{\mu\nu}$. Remember the program we had for E&M – we replaced an explicit source coupling with a change in derivative definition. Here, that shift is implicit – the covariant derivative, which we must use in order to make scalar actions, is modified by the presence of Einstein's equation. That is – we do not know *a priori* what sort of space-time we have. All we do know is that there is a metric space, so the metric is connected to the $\Gamma^{\alpha}_{\ \beta\gamma}$ in the usual way.

Consider our scalar field theory,

$$S_\phi = -\int d\tau\, \sqrt{-g}\, \phi_{,\mu}\, \phi_{,\nu}\, g^{\mu\nu} \tag{27.8}$$

then varying w.r.t. ϕ gives:

$$\delta S_\phi = \int d\tau\, \left(2\,\sqrt{-g}\, \phi_{,\nu}\, g^{\mu\nu}\right)_{,\mu}\, \delta\phi, \tag{27.9}$$

and we can use the property of densities: $(\sqrt{-g}\, f^{\alpha})_{,\alpha} = (\sqrt{-g}\, f^{\alpha})_{;\alpha}$ to rewrite the integrand in terms of the covariant derivative – noting that $g_{\mu\nu;\alpha} = 0$ gives

$$\delta S_\phi = \int d\tau\, \left(2\,\sqrt{-g}\, \phi_{,\nu;\mu}\, g^{\mu\nu}\right)\, \delta\phi. \tag{27.10}$$

Now to get a stationary S, we set the integrand to zero – as a point of notation, $\phi_{,\nu;\mu} = \phi_{;\nu\mu}$ for scalars, and we have the field equation

$$\phi_{;\mu}{}^{\mu} = 0 \tag{27.11}$$

just the generalized Laplacian defined by our metric.

That's good, but what does the scalar do to $g_{\mu\nu}$? We must always keep in mind that tacked on to any field theory is GR, so that the above action should really be written (using α as a coupling constant):

$$S = -\alpha \int d\tau \sqrt{-g}\, \phi_{,\mu}\, \phi_{,\nu}\, g^{\mu\nu} + \int d\tau \sqrt{-g}\, g^{\mu\nu}\, R_{\mu\nu}. \tag{27.12}$$

This doesn't change the variation w.r.t. ϕ, but we must also vary w.r.t. $g^{\mu\nu}$. Varying with respect to the metric is precisely how the stress tensor of a field theory is developed – we are making the theory coordinate-invariant. The GR action varies in the usual way for $\delta g^{\mu\nu}$, providing the left-hand side of Einstein's equation: $R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R$. But now we have to vary the metric sitting in the first term above, the "matter" term.

The only minor difference here, and it is a feature of the construction of S_ϕ, is that we have to take the derivative of the Lagrangian w.r.t. $g^{\mu\nu}$ rather than the covariant form. In terms of this, the stress tensor is defined to be

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = 2\, \frac{\partial \bar{\mathcal{L}}}{\partial g^{\mu\nu}} - g_{\mu\nu}\, \bar{\mathcal{L}}. \tag{27.13}$$

For our scalar field, $\bar{\mathcal{L}} = \phi_{,\mu}\, \phi_{,\nu}\, g^{\mu\nu}$, and the stress tensor is trivial:

$$T_{\mu\nu} = \alpha \left(2\, \phi_{;\mu}\, \phi_{;\nu} - g_{\mu\nu}\, \phi_{;\alpha}\, \phi_{;}{}^{\alpha}\right), \tag{27.14}$$

as usual. It is easy to see that with the field equation for ϕ in place, we have $T^{\mu\nu}{}_{;\nu} = 0$ so that this stress-tensor is properly conserved. We have, finally, the full variation giving:

$$\boxed{R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R - T_{\mu\nu} = 0,} \tag{27.15}$$

which holds for any $T_{\mu\nu}$ we might cook up. In the end α can be used to set the final units.

This, then, is Einstein's equation – we take a stress-energy tensor $T_{\mu\nu}$ and use this to determine $g_{\mu\nu}$ from the combination of metric and its derivatives (and second derivatives) found in the Einstein tensor. Don't be fooled by the "simplicity" of $G_{\mu\nu} = T_{\mu\nu}$, this set of ten PDEs are not easy to solve in general. Only very specific cases are tractable, and there are many configurations whose solutions are still unknown – at more general levels, numerical solutions are the only available option.

As a final note, we review the prescription for coupling an external theory (like E&M, say) to gravity – you take the free field form of the action, define the fields consistently and leave naked metrics to close indices. Then, you replace commas with semicolons and vary normally. The sourcing of gravity by the external theory is automatic from the metric.

For example, if we want to couple E&M to gravity, we would take the Lagrangian:

$$\bar{\mathcal{L}} = F^{\mu\nu}\,(A_{\nu,\mu} - A_{\mu,\nu}) - \frac{1}{2}\,F^{\mu\nu}\,g_{\alpha\mu}\,g_{\beta\nu}\,F^{\alpha\beta}. \tag{27.16}$$

Now Maxwell's equations will be in terms of covariant derivatives, and the stress tensor of E&M will source the gravitational field $g_{\mu\nu}$. Because of the presence of a metric in any Lagrangian (we must use the metric to form scalars), *everything* couples to gravity – any Lagrangian we write down will provide a source, since $\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \neq 0$. This is the mathematical manifestation of the idea that all forms of energy (mass and otherwise) have an effect on the geometry of space-time.

27.3 Source-Free Solutions

Now that we have seen how matter couples, we will retreat to the more tractable vacuum equations: $R_{\mu\nu} = 0$. In electrodynamics, we solve the source-free problem almost immediately – the Coulomb field is defined everywhere (except at the origin), and satisfies $\nabla^2 V = 0$ for $r \neq 0$. This spherically symmetric field is the simplest possible solution to Laplace's equation (except for $V = 0$). Of course, because we have a delta function source, we can build up solutions for more complicated distributions. That's the role of superposition, an observational fact that is included in Maxwell's equations (they are linear).

In general relativity, the situation is quite different – Einstein's equation is nonlinear, so we do not expect superposition to hold (it doesn't). Even the simple spherically symmetric analogue of the Coulomb field of E&M is not immediately apparent. We will have to solve $R_{\mu\nu} = 0$ for a spherically symmetric metric, and once we have the solution, we can do "nothing" with it (in the sense of building other solutions).

The salvation is that there is really only one astrophysically interesting solution to $R_{\mu\nu} = 0$, the metric appropriate to rotating, spherical massive bodies, and the Coulomb solution is then the slow-motion limit of this. So we will spend some time with the "Schwarzschild" metric understanding what it has to say above and beyond Newtonian gravity.