Linearized Gravity

Lecture 28

Physics 411
Classical Mechanics II

November 7th, 2007

We have seen, in disguised form, the equations of linearized gravity. Now we will pick a gauge for our linearized field equations. As with E&M, this gives us a way to discuss the physically relevant portion of the solutions as opposed to leaving an infinite family. For the spherically symmetric solution we find from the vacuum field equations, we will gain, through linearization, an interpretation for the functions scaling dt and dr.

In addition, we will learn that there are two separate sources for perturbations to Minkowski – the usual Newtonian scalar potential, and a new vector potential associated with moving mass (or more generally, moving energy density). By considering the (linearized) geodesic equations, we can associate these naturally with the scalar and magnetic vector potential from E&M.

28.1 Return to Linearized Field Equations

The linearized field equations can be obtained by writing $G_{\mu\nu}$ in terms of a metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Alternatively, we can simply take the field equations we got when we considered the most general action for a second rank, symmetric field theory. This latter point of view enforces the idea that the linear field equations are really meant to be interpreted as field equations on an explicitly Minkowski background. In trace-reversed form, they read:

$$\tilde{G}_{\mu\nu} = \partial^\rho \, \partial_{(\mu} \bar{h}_{\nu)\rho} - \frac{1}{2} \, \partial_\rho \partial^\rho \, \bar{h}_{\mu\nu} - \frac{1}{2} \, \eta_{\mu\nu} \, \partial^\alpha \partial^\beta \, \bar{h}_{\alpha\beta}, \tag{28.1}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \, \eta_{\mu\nu} \, h$, and $h_{\mu\nu}$ the metric perturbation: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We are going to use our gauge invariance, $h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + f_{(\mu,\nu)}$

(analagous to $A_\mu \to A_\mu + f_{,\mu}$ in E&M) to get rid of the divergence terms above, that is, we want $\partial^\rho \bar{h}'_{\nu\rho} = 0$,

$$\begin{aligned}\partial^\rho \bar{h}'_{\nu\rho} &= \partial^\rho \left(h_{\nu\rho} - \frac{1}{2}\,\eta_{\nu\rho}\,h + \partial_{(\nu} f_{\rho)} - \frac{1}{2}\,\eta_{\nu\rho}\,\partial^\alpha\,f_\alpha \right) \\ &= \partial^\rho\,\bar{h}_{\nu\rho} + \frac{1}{2}\,\partial^\rho\partial_\nu\,f_\rho + \frac{1}{2}\,\partial^\rho\partial_\rho\,f_\nu - \frac{1}{2}\,\partial_\nu\partial^\alpha\,f_\alpha \\ &= \partial^\rho\,\bar{h}_{\nu\rho} + \frac{1}{2}\,\partial^\rho\partial_\rho\,f_\nu.\end{aligned} \tag{28.2}$$

To get this equal to zero, we need to solve Poisson's equation for f_ν: $\partial^\rho\partial_\rho\,f_\nu = -2\,\partial^\rho\,\bar{h}_{\nu\rho}$. Suppose we do that (Lorentz gauge in E&M), then the (really final) linearized Einstein tensor reads:

$$\boxed{\tilde{G}'_{\mu\nu} = -\frac{1}{2}\,\partial_\rho\partial^\rho\,\bar{h}'_{\mu\nu}} \tag{28.3}$$

where the primes remind us that we have chosen a gauge and transformed already. This is an interesting equation – it says among other things that in source free regions, the metric perturbation $\bar{h}'_{\mu\nu}$ can form waves (from now on, I drop the primes indicating the transformation, but remember the gauge condition). We will return to that later on, for now I want to focus on a matter source of some variety, so we have to think about the right-hand side of Einstein's equation in a weak limit.

What should we choose as the form for the matter generating the metric perturbation? Let's consider rigid-body sources – here we mean that the internal stresses, the T_{ij} components of the stress tensor, are zero. That's not strictly speaking possible – even for non-interacting dust at rest, we can boost to a frame in which there are diagonal components aside from T_{00}, but we are taking these to be small (they have $(v/c)^2$ factors associated with them). We have a mass density and a rigid velocity, so generically, the stress tensor takes the form:

$$T_{\mu\nu} \doteq \begin{pmatrix} \rho & -j_1 & -j_2 & -j_3 \\ -j_1 & 0 & 0 & 0 \\ -j_2 & 0 & 0 & 0 \\ -j_3 & 0 & 0 & 0 \end{pmatrix} \tag{28.4}$$

(remember that it is $T^{\mu\nu}$ that we are used to, the negatives come from lowering using Minkowski) and Einstein's equation in matrix form looks like:

$$\partial^\alpha \partial_\alpha \begin{pmatrix} \bar{h}_{00} & \bar{h}_{01} & \bar{h}_{02} & \bar{h}_{03} \\ \bar{h}_{01} & \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ \bar{h}_{02} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{23} \\ \bar{h}_{03} & \bar{h}_{13} & \bar{h}_{23} & \bar{h}_{33} \end{pmatrix} = -16\,\pi \begin{pmatrix} \rho & -j_1 & -j_2 & -j_3 \\ -j_1 & 0 & 0 & 0 \\ -j_2 & 0 & 0 & 0 \\ -j_3 & 0 & 0 & 0 \end{pmatrix} \tag{28.5}$$

(I have dropped an overall ϵ on the left, and we assume, on the right as well – the sources must be small) all the spatial-spatial terms on the right vanish by assumption, so the spatial-spatial components of $\bar{h}$ must satisfy

$$\partial^\alpha \partial_\alpha \bar{h}_{ij} = 0 \rightarrow \nabla^2 \bar{h}_{ij} = 0, \tag{28.6}$$

the time derivatives of the above are small if the motion of the *source* is small (compared to the speed of light[1]) and I have discarded them. But consider the boundary conditions, as $r \rightarrow \infty$, we want the metric perturbation to vanish so that we are left with normal Minkowski space-time very far away from the source. A Laplacian can have no minima or maxima on the interior, so we conclude that the spatial components $\bar{h}_{ij}$ are identically zero.

Moving on to the less trivial components. The time-time and time-spatial equations read:

$$\boxed{\begin{aligned} \nabla^2 \bar{h}_{00} &= -16\,\pi\,\rho \rightarrow \nabla^2 \phi = -4\,\pi\,\rho \text{ with } \phi \equiv \frac{h_{00}}{2} \\ \nabla^2 \bar{h}_{0i} &= 16\,\pi\,j_i \rightarrow \nabla^2 \mathbf{A} = -4\,\pi\,\mathbf{j} \text{ with } A_i \equiv -\frac{h_{0i}}{4} \end{aligned}} \tag{28.8}$$

(keeping in mind that $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\,\eta_{\mu\nu}\,\bar{h}$). This is "just" the definition of the electromagnetic four-potential (in Lorentz gauge) in terms of sources ρ and $\mathbf{j}$. Here they are mass distribution and "mass current" distribution (this says that in GR, a moving mass generates a current just like a moving charge does in E&M).

We must connect these to motion – what we want is effectively the force equation $\mathbf{F} = m\mathbf{a}$ – but practically, what we will do is generate it from the geodesic equation of motion for a test particle. So, referring to our classical

[1]With units, we have $dx^i = v^i\,dt = \frac{v^i}{c}\,dx^0 = \epsilon^i\,dx^0$ and then

$$\epsilon\,\frac{\partial \bar{h}}{\partial x^i} \sim \frac{\partial \bar{h}}{\partial x^0}. \tag{28.7}$$

intuition and our geodesic discussions, we know that the Lagrangian for a test particle in GR is:

$$L = \frac{1}{2}\, g_{\mu\nu}\, \dot{x}^\mu\, \dot{x}^\nu \tag{28.9}$$

varying this, we get equations of motion that look like:

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\, \dot{x}^\beta\, \dot{x}^\gamma = 0 \tag{28.10}$$

and what we'll do is move the Γ over to the right and interpret it as a force. Remember that our dots here refer to the proper time τ of the particle, but if we assume the particle is moving slowly, the proper time and coordinate time t coincide (to within our ϵ error), so we will just view $\dot{x}$ as $\frac{d}{dt}x$ etc. These velocities are themselves small (remember that a time derivative is $\epsilon\times$"spatial derivatives") so we will also drop terms quadratic in the spatial velocity. That leaves us relatively few terms to consider – we set $\alpha = i$ in order to look at the spatial components.

$$\Gamma^i{}_{\beta\gamma}\, \dot{x}^\beta\, \dot{x}^\gamma = \frac{1}{2}\, \eta^{i\mu}\, (h_{\beta\mu,\gamma} + h_{\gamma\mu,\beta} - h_{\beta\gamma,\mu})\, \dot{x}^\beta\, \dot{x}^\gamma, \tag{28.11}$$

but because of the diagonal form of the Minkowski metric $\eta^{i\mu} = \eta^{ij}$ (it's actually just δ^i_j, but let me leave it in this form for now). Once again, because temporal derivatives go as epsilon times spatial ones (now applied to the slow motion of the *particle*), we will drop terms quadratic in the velocities from the above, and set time-derivatives of the metric to zero. With these approximations, the sum looks like:

$$\Gamma^i{}_{\beta\gamma}\, \dot{x}^\beta\, \dot{x}^\gamma = \frac{1}{2}\, \dot{x}^0\, \dot{x}^0\, \eta^{ij}\, (-h_{00,j}) + \frac{1}{2}\, \dot{x}^0\, \dot{x}^k\, \eta^{ij}\, (h_{0j,k} - h_{0k,j}) + \frac{1}{2}\, \dot{x}^k\, \dot{x}^0\, \eta^{ij}\, (h_{0j,k} - h_{k0,j}), \tag{28.12}$$

with $\dot{x}^0 = 1$ in these units, we put this into the equation of motion to get:

$$\begin{aligned} \ddot{x}^i &= -\left(-\frac{1}{2}\, \partial^i\, h_{00} + \dot{x}^k\, \eta^{ij}\, (\partial_k\, h_{0j} - \partial_j\, h_{0k}) \right) \\ &= \frac{1}{2}\, \nabla\, h_{00} + \mathbf{v} \times (\nabla \times \mathbf{h}_0) \end{aligned} \tag{28.13}$$

where I'm writing in the usual Cartesian vector notation. Finally, we have from (28.8) the connection to what we think of as potentials, the equation of motion for a test particle in linearized gravity can be written,

$$\boxed{\ddot{\mathbf{x}} = \nabla\, \phi - 4\, \mathbf{v} \times (\nabla \times \mathbf{A})} \tag{28.14}$$

with the potentials determined by the source according to (28.8).

There's no real way for me to ensure that balloons fall from the ceiling at the moment you see the above equation, which is too bad. We've taken a full tensor theory, linearized and massaged it into a vector theory which is precisely E&M with the wrong signs (the potential for Newtonian gravity is opposite E&M) and a counting factor of four. Indeed, it answers a question that everyone has in E&M – when you look at how close the mass potential is to the electromagnetic one, you imagine that it is possible to write an electro-magneto-static-like theory – but then shouldn't there be an analogue of **B** for moving masses? Classically, this is not the case, but we see here that GR predicts a gravitational interaction with mass "currents".