Orbital Motion in Schwarzschild Geometry

Lecture 29

Physics 411
Classical Mechanics II

November 9th, 2007

We have seen, through the study of the weak field solutions of Einstein's equation that in fact, we can, to first order replace GR with Newtonian gravity. What was more interesting there was the second order effect – a new "force of nature" akin to a magnetic current generated by a magnetic vector potential $\mathbf{A}$ that coupled to test particles in the familiar way $\mathbf{F} = m\,\mathbf{v} \times \mathbf{B}$.

Of course, there is no new force of nature here, only what looks like one from the point of view of a flat background – in reality, this "mass current" coupling is nothing more than a perturbative effect – a correction to geodesic motion that comes from linearization of a general metric.

Going back, we want to work directly from the full (nonlinear) Schwarzschild metric written in our usual Schwarzschild coordinates and with the interpretation we have developed from weak field approximation.

29.1 Recall

Remember the situation in Newtonian gravity – we had a Lagrangian, three-dimensional of course, and a potential. Using the Lagrange approach, we recovered an equation of motion $\rho(\phi)$ (where $\rho = 1/r$ the inverse of the radial coordinate). Alternatively, we looked at the Hamiltonian, identified constants of the motion from Killing vectors and used those to develop the identical equation.

Either way, we end up with a second order oscillator ODE:

$$\rho''(\phi) = -\rho(\phi) + \frac{M}{J_z^2} \tag{29.1}$$

the solutions of which give us back elliptical orbits.

So be on the lookout for this equation to come back unchanged as a zeroth order effect. What sort of physics do we expect from a perturbation to this? Well, almost any spherically symmetric perturbation induces a "precession" (of both the perihelion and aphelion), a motion of the closest approach – it shifts itself.

29.2 Test Particles in Schwarzschild Geometry

Now we begin the process of understanding the Schwarzschild solution by the introduction of test particles. We have the metric form, and we know the geodesic equation for a point particle, if we drop it in somewhere, what happens to it? There are a two interesting options: it could "fall", or it could "orbit". Let's take the orbiting case first, since this provides the celebrated perihelion precession result that was one of the first successes of Einstein's theory. We have also built up a lot of machinery for understanding orbital solutions in Newtonian Gravity, and there are nice parallels there.

We have the Lagrangian

$$L = \frac{1}{2}\,\dot{x}^\alpha\,\dot{x}^\beta\,g_{\alpha\beta}, \tag{29.2}$$

for a particle in a gravitational field described by $g_{\alpha\beta}$ (Schwarzschild). There is no potential, so the Lagrangian is equal in value to the Hamiltonian. One of the first things we can do is describe, for example, all the Killing vectors of the space-time. It is interesting that the spatial Killing vectors of Schwarzschild are precisely those of Newtonian gravity (a consequence of our assumed spherical symmetry), all three components of angular momentum (defined in the usual spherical way) lead to conserved quantities, and we can again set motion in a plane by choosing (J_x, J_y) appropriately. The axis of the orbit is again aligned with the $\theta = 0$ line, and we have one component of angular momentum left: J_z. The problem is that since we are now in four dimensions, we need one extra constant of the motion. We have it automatically, but it takes a moment to see – the issue is that while we have our Lagrangian and our Hamiltonian $H = \frac{1}{2}\,p_\alpha\,p_\beta\,g^{\alpha\beta}$, it is no longer the case that $H = E$ the "energy" of the system. H (and L) are still conserved, of course, but their value is somewhat arbitrary, reflecting more about what we mean by parametrization than anything else.

Let me be clear, all we are doing is finding the geodesics of the Schwarzschild geometry, the geodesic equation is really

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{29.3}$$

and by choosing τ in our parametrization $x^\alpha(\tau)$ we are just setting the unit of measurement. For "proper time", the geodesic is in affine parametrization and we write $ds^2 = d\tau^2$ (again, with $c = 1$). The Hamiltonian then is just the four-momentum invariant: in Minkowski space-time, the relativistic relation is $p_\alpha\, p^\alpha = -E^2 + \vec{p} \cdot \vec{p} = -m^2\, c^4 = -1$ (in our units). This tells us that, as usual, it is time-translation-invariance that sets the energy. If we look at the quantity conserved under coordinate time translation, it's just the t momentum (since t is an ignorable coordinate), so that:

$$p_t = \frac{\partial}{\partial \dot{t}}\, L = -\left(1 - \frac{2\, M}{r}\right)\, \dot{t} \equiv -E. \tag{29.4}$$

Once the motion is set in the $\theta = \frac{\pi}{2}$ plane, we have the constants E, J_z and $H = L = -\frac{1}{2}$, this last I'm setting as a matter of convenience. Time-like particles, in affine parametrization (proper time) have $d\tau^2 = -1$ and so $H = L = -\frac{1}{2}$.

Of course, solving for the Killing vectors tells you this immediately – you get the angular momentum ones and one that looks like $f^\alpha \doteq \delta^\alpha_0$. In that setting, the conservation of H itself is described via a second-rank Killing tensor, also known as the metric (the ultimate Killing tensor, since $g_{\mu\nu;\alpha} = 0$) with the usual interpretation of τ-translation (the transformation generated by the metric takes the particle along its geodesic from τ to $\tau + d\tau$).

With that potentially confusing digression, we get back to solving the problem. Basically, we will do this in a manner identical to our work on orbits in Newtonian fields, so the first step is to introduce a new radial coordinate $\rho \equiv \frac{1}{r}$ inducing the transformed metric

$$g_{\mu\nu} \doteq \begin{pmatrix} -(1 - 2\, M\, \rho) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho^4\, (1 - 2\, M\, \rho)} & 0 & 0 \\ 0 & 0 & \rho^{-2} & 0 \\ 0 & 0 & 0 & \rho^{-2}\, \sin^2\theta \end{pmatrix}. \tag{29.5}$$

Next we identify the J_z angular momentum – we know the answer, but it can be trivially obtained (again) by calculating the p_ϕ momentum,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}\, \rho^{-2}\, \sin^2\theta \equiv J_z \tag{29.6}$$

and set $\theta = \frac{1}{2}\pi$, $\dot{\theta} = 0$ corresponding to choices for J_x and J_y. With two additional constants E and $H = -\frac{1}{2}$, we can reduce the entire problem to an equation involving a single derivative of ρ:

$$\begin{aligned} -\frac{1}{2} = L &= \frac{1}{2}\left(J_z^2\,\rho^2 - \frac{E^2\,\rho^4 - \dot{\rho}^2}{\rho^4\,(1-2\,M\,\rho)}\right) \\ &\downarrow \\ \dot{\rho}^2 &= \rho^4\left(E^2 - (1-2\,M\,\rho)\,(1+J_z^2\,\rho^2)\right) \end{aligned} \tag{29.7}$$

and now as "usual", for bound orbits, we will transform to $\rho(t) \rightarrow \rho(\phi)$, under which $\dot{\rho} = \frac{d\rho}{d\phi}\frac{d\phi}{dt} = \rho'\,J_z\,\rho^2$, then we can write

$$\boxed{(\rho')^2 = \left(\frac{(E^2-1)}{J_z^2} + \frac{2\,M\,\rho}{J_z^2} - \rho^2 + 2\,M\,\rho^3\right).} \tag{29.8}$$

So far, our approach has been similar to the Hamiltonian discussion of our earlier work on Newtonian gravity – the problem is, rather than something like the effective Newtonian potential which is quadratic in r (and ρ, for that matter), the above is cubic. We can proceed by factoring the cubic and identifying the turning points of the radial equation, setting one equal to the perihelion, the other to aphelion, and ignoring a spurious extra root. But our goal is to predict the perturbation in the orbits of planets due to GR effects, and the perturbation parameter will be M. But this leads to a degenerate perturbation (as M goes to zero, the above cubic loses a root, so there is a discontinuity in the space of solutions even for small values of M), it turns out to be easier to treat this situation by taking a ϕ derivative on both sides, this gives us an equation comparable to (29.1). We lose constants of integration, and will have to put them back as appropriate, but this is a reasonable price to pay to keep the perturbation analysis simple. Differentiating gives:

$$\boxed{\rho'' = -\rho + \frac{M}{J_z^2} + 3\,M\,\rho^2.} \tag{29.9}$$

The term on the left and first two terms on the right are familiar from (29.1), but now there is a new term $3\,M\,\rho^2$. Let me pause here before going further, (29.9) is exact, you can solve it formally with (the aptly named) elliptic integrals, or numerically. Both options are important for realistic GR calculations. What we are about to do is linearize, and hence, specialize our approach.

Take a perturbative solution: $\rho = \tilde{A} + B\,\tilde{\rho}(\phi)$, and expand in powers of B – that is, we are assuming the orbit we want to describe is close to a circle $B = 0$. Then we have, defining $A \equiv \frac{M}{J_z^2}$, inserting our ρ into (29.9), and collecting in powers of B:

$$\begin{aligned} B^0 &: \tilde{A} - 3\,\tilde{A}^2\,M - A = 0 \Rightarrow \tilde{A} = \frac{1 - \sqrt{1 - 12\,A\,M}}{6\,M} \Rightarrow \tilde{A} \approx A + O(A^2 M) \\ B^1 &: \tilde{\rho}'' = -\tilde{\rho}\,(1 - 6\,\tilde{A}\,M) \Rightarrow \tilde{\rho} = \cos\left(\phi\sqrt{1 - 6\,\tilde{A}\,M}\right) \Rightarrow \tilde{\rho} \approx \cos(\phi(1 - 3\,A\,M))\,, \end{aligned} \tag{29.10}$$

where the approximations come in assuming that the effect of the additional term AM is small, as indeed it must be – for most planetary orbits, we see only very small corrections to Keplerian ellipses. Our approximate solution reads:

$$\rho(\phi) \approx A + B\,\cos(\phi(1 - 3\,A\,M)) \Rightarrow r(\phi) \equiv \frac{1}{\rho(\phi)} = \frac{1}{A + B\,\cos(\phi\,(1 - 3\,A\,M))}. \tag{29.11}$$

29.2.1 Precession

Before finishing the job, let us look at the sorts of trajectories we can get out of this type of geometric equation – again, we identify A and B as describing the radial turning points – but now we have an additional parameter δ:

$$r(\phi) = \frac{1}{A + B\,\cos(\phi\,(1 - \delta))}, \tag{29.12}$$

A tells us the lateral extent of the orbit, and B tells us the "circularity", while δ represents a phase shift of sorts– if we start at $\phi = 0$, $r = 1/(A + B)$, and go $2\,\pi$ in ϕ, we will not end up back at $r = 1/(A + B)$, but rather will end up at $r = 1/(A + B\,\cos(2\,\pi\,\delta))$. In order to get back to $r = 1/(A + B)$, the starting point, we must go $\bar{\phi} \equiv 2\,\pi\,(1 - \delta)$ radians. The observational effect of all of this is shown in Figure 29.1.

Now for the perturbation coming from GR, we have from (29.11): $\delta = 3\,A\,M$. For δ small, $\bar{\phi} \approx 2\,\pi\,(1 + \delta)$, so the advance (since $\bar{\phi} > 2\pi$) per orbit is $\Delta\phi \equiv \bar{\phi} - 2\,\pi = 2\,\pi\,\delta = 2\,\pi\,(3\,A\,M)$. A refers to an orbital feature – typically, planetary orbits are described in terms of their eccentricity e and

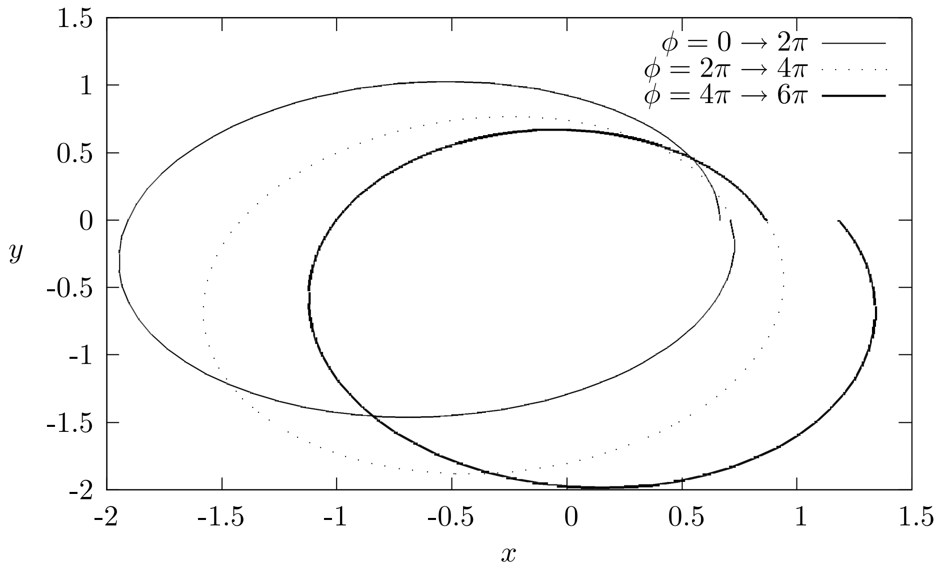

Figure 29.1: First three "orbits" of a precessing ellipse $r = \frac{1}{A+B\,\cos(\phi\,(1-\delta))}$ with $A = 1$, $B = 0.5$ and $\delta = .1$.

semi-major axis a related to our A and B variables by: $e = B/A$, and

$$\frac{1}{2}\left(\frac{1}{A\,(1+e)} + \frac{1}{A\,(1-e)}\right) = a \Rightarrow A = \frac{1}{a\,(1-e^2)}. \tag{29.13}$$

So we have $\Delta\phi = 6\,\pi\,M/(a\,(1-e^2))$ per orbit. Because it is a small effect, we accumulate this over 100 years – the observable quantity is:

$$\frac{\Delta\phi}{T} \times \frac{100 \text{ years}}{\text{century}} = \frac{6\,\pi\,M}{T\,a\,(1-e^2)} \times \frac{100 \text{ years}}{\text{century}}, \tag{29.14}$$

where a is the semi-major axis of the orbit, e its eccentricity, T its period in years, and M must be measured in units of length. Fortunately, people do measure M in units of length, and we can either look it up or generate it using appropriate constants.

For Mercury, which, as the closest planet, exhibits the largest effect, we can go to the tables and find $a = 57.91 \times 10^6$ km, $e = .2056$, $T = .24084$ years, and the mass of the sun is given by

$$\begin{aligned} M_{km} \equiv \frac{M_{kg}\,G}{c^2} &= \frac{\left(2\times 10^{30} \text{ kg}\right)\left(6.672\times 10^{-11} \text{ m}^3/(\text{kg s}^2)\right)\left(\frac{1 \text{ km}}{1000 \text{ m}}\right)}{\left(3\times 10^8 \text{ m/s}\right)^2} \\ &= 1.483 \text{ km}, \end{aligned} \tag{29.15}$$

so back in (29.14), we have

$$\boxed{\begin{aligned} \frac{\Delta\phi}{T} \times \frac{100 \text{ years}}{\text{century}} &= \frac{6\pi(1.483 \text{ km})}{(57.91\times 10^6 \text{ km})(1-(.2056)^2)\,(.24084 \text{ years})} \times \frac{100 \text{ years}}{\text{century}} \\ &= 2.093\times 10^{-4}\left(\frac{360^{\text{deg}}}{2\pi}\right)\left(\frac{60'}{1^{\text{deg}}}\right)\left(\frac{60''}{1'}\right)\frac{1}{\text{century}} \\ &\approx 43''/\text{century}, \end{aligned}} \tag{29.16}$$

which agrees well with the "observational" result (this is a tricky thing to observe) of $42.56'' \pm .94''$.

As an *a posteriori* check of the validity of our perturbation calculation, let me go back and calculate A in its natural units (1/length), from (29.13), and for Mercury's orbital parameters. We have:

$$A = \frac{1}{a\,(1-e^2)} = \frac{1}{(57.91\times 10^6 \text{ km})(1-.2056^2)} = 1.8\times 10^{-8}\,\frac{1}{\text{km}}, \tag{29.17}$$

and the claim, in for example (29.10), is that $A^2 M$ is small – we've kept terms of order $A\,M = (1.8 \times 10^{-8}) \times 1.483 \approx 2.67 \times 10^{-8}$, but then $A^2 M \approx 4.8 \times 10^{-16}$ 1/km! Our other approximation was that B should also be small – $B = A\,e = 1.8 \times 10^{-8}$1/km $\times .2056 \approx 3.7 \times 10^{-9}$ 1/km. We've kept terms of order B, and again, the corrections to this are order $B^2 \approx 10^{-16}$ 1/km^2. Evidently, our expansion is justified.

This is the famous first test of general relativity, one which Einstein worked out in his paper introducing the subject. It is a beautiful result, and especially so because almost any perturbation to an elliptical orbit causes precession to first order – that he got just the right form to account for the precession is shocking. It is also important, historically, that because everything perturbing an ellipse makes it precess, the actual observed precession of Mercury is 5000" per century, but for decades previous to Einstein's paper, people had been whittling down the excess by doing precise calculations on the perturbative effects of the other planets in the solar system, comets, etc. One imagines that had the GR calculation been done a century earlier, it would not have seemed so great.

29.3 Exact Solutions

With our linearized approximation, it is tempting to close the book on massive bodies "falling" (along geodesics) in the Schwarzschild geometry. There are a number of issues left to consider, not the least of which is: for the full equations of motion, what happens if we are not in a "linearized" regime? This is a question which can be answered by proceeding with the linearization to higher and higher order terms.

At some point, this procedure becomes tedious (depends on your stamina), and we might imagine moving to a purely computational solution. This is easier done than said – the ODEs we get are exceedingly well-behaved, both from a function-theoretic point of view and as a point of numerical analysis.

What is the difference between an exact solution and a perturbative one? How much more exotic can the "full" (numerically exact, anyway) solution get? In Schwarzschild geometry, this is a difficult question to answer, we can *look* at the orbits, measure their precession, etc. But the general observation is that not much new (aside from precession) is going on.

In more complicated situations, as we shall see, there are far more exotic trajectories available. In particular, again taking the subset of GR that is of astrophysical interest, one can get "zhoom-whirl" trajectories, in addition to precession effects outside of the equatorial plane.