Orbital Motion

Lecture 3

Physics 411
Classical Mechanics II

August 31st 2007

After the indexing tour-de-force from last time, we are still stuck solving the problem of orbital motion. The point of that introduction to the covariant formulation of the equations of motion will become clear as we proceed. Today we take a step back, dangling indices do not help to solve the problem in this case. Although it is worth noting that we have taken coordinate independence to a new high (low?) with our metric notation – now you *really* don't know what coordinate system you are in.

So we will solve the equations of motion in a chosen set of coordinates – the standard ones. There is a point to the whole procedure – GR is a coordinate independent theory, we will primarily write statements that look a lot like what we saw last time, *but*, in order to "solve" a problem, we will always have to introduce coordinates. That is the plan for today. After we have dispensed with Keplerian orbits, we will move on and solve the exact same problem using the Hamiltonian formulation, and for that we will need to discuss vectors and tensors again.

3.1 Ellipse

Going back to the Lagrangian for this problem, in abstract language, we had:

$$L = \frac{1}{2} m \, \dot{x}^\mu \, g_{\mu\nu} \, \dot{x}^\nu - U(r). \tag{3.1}$$

Suppose we start off in spherical coordinates, so that we know the metric is

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{3.2}$$

with coordinate differential $dx^\alpha \doteq (dr, d\theta, d\phi)$.

We will transform one more time: let $\rho = r^{-1}$, then the metric (specified equivalently by the line element) becomes:

$$\begin{aligned} ds^2 &= dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2 \\ &= \frac{1}{\rho^4}\, d\rho + \frac{1}{\rho^2}\, d\theta^2 + \frac{1}{\rho^2}\, \sin^2\theta\, d\phi^2. \end{aligned} \tag{3.3}$$

In matrix form, this reads:

$$g_{\mu\nu} \doteq \begin{pmatrix} \frac{1}{\rho^4} & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2}\sin^2\theta \end{pmatrix} \tag{3.4}$$

with the new coordinate differential $dx^\alpha \doteq (d\rho, d\theta, d\phi)$.

The potential is spherically symmetric, meaning that there are no preferred directions, functionally, that it depends only on r. We can *set* $\theta = \frac{\pi}{2}$ and $\dot\theta = 0$ to put the motion in a specific plane (the horizontal plane – for Cartesian coordinates in their standard configuration, this is the $x-y$ plane). Now that's all well and good, but we have to be careful – when we use information about a solution prior to variation, we can lose the full dynamics of the system. As an example, consider a free particle classical Lagrangian – just $L_f = \frac{1}{2}\, m\, \dot x^2$ – we know that the solutions to this are vectors of the form $\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}\, t$. If we put this into the Lagrangian, we get $L_f = \frac{1}{2}\, m\, v^2$, just a number. We cannot vary a number and recover the equations of motion, so we have lost all dynamical information by introducing, in this case, the solution from the equations of motion themselves. That may seem obvious, but we have done precisely this in the above specialization to planar motion. In this case, it works out okay, but you might ask yourself why you can't equally well take the motion to lie in the $\theta = 0$ plane? We will address this later on when we discuss the Hamiltonian.

Putting $\theta = \frac{\pi}{2}$ reduces the dimensionality of the problem. We may now consider a two-dimensional metric with $dx^\alpha \doteq (d\rho, d\phi)$, and

$$g_{\mu\nu} \doteq \begin{pmatrix} \frac{1}{\rho^4} & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}. \tag{3.5}$$

Our Lagrangian in these coordinates reads:

$$\boxed{L = \frac{1}{2}\, m \left(\frac{\dot\rho^2}{\rho^4} + \frac{\dot\phi^2}{\rho^2} \right) - U(\rho).} \tag{3.6}$$

At this point, we could play all the usual games, but it is easiest to note that there is no ϕ dependence in the above, i.e. ϕ is an ignorable coordinate. From the equation of motion, then:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \tag{3.7}$$

we learn that the momentum conjugate to ϕ (defined by $\frac{\partial L}{\partial \dot{\phi}}$) is conserved. So we substitute a constant for $\dot{\phi}$:

$$\frac{\partial L}{\partial \dot{\phi}} = J_z = \frac{m\,\dot{\phi}}{\rho^2} \longrightarrow \boxed{\dot{\phi} = \frac{J_z}{m}\,\rho^2.} \tag{3.8}$$

Now take the equation of motion for ρ, substituting the $J_z \rho^2/m$ for $\dot{\phi}$

$$\begin{aligned}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho}\right)\Bigg|_{\dot{\phi}=\frac{J_z}{m}\rho^2} &= m\,\frac{d}{dt}\left(\frac{\dot{\rho}}{\rho^4}\right) + 2\,m\,\dot{\rho}^2\rho^{-5} + \frac{J_z^2}{m}\,\rho + \frac{dU}{d\rho} \\ &= m\,\frac{\ddot{\rho}}{\rho^4} - 4\,m\,\frac{\dot{\rho}^2}{\rho^5} + 2\,m\,\dot{\rho}^2\,\rho^{-5} + \frac{J_z^2}{m}\,\rho + \frac{dU}{d\rho} \\ &= m\,\frac{\ddot{\rho}}{\rho^4} - 2\,m\,\frac{\dot{\rho}^2}{\rho^5} + \frac{J_z^2\,\rho}{m} + \frac{dU}{d\rho}.\end{aligned} \tag{3.9}$$

We can reparametrize – rather than finding the time development of the $\rho(t)$ and $\phi(t)$ coordinates, the geometry of the solution can be uncovered with $\rho(\phi)$ – so note that:

$$\begin{aligned}\dot{\rho} &= \rho'\,\dot{\phi} = \rho'\,\frac{J_z}{m}\,\rho^2 \\ \ddot{\rho} &= \rho''\,\dot{\phi}\,\frac{J_z}{m}\,\rho^2 + \rho'\,\frac{J_z}{m}\,2\,\rho\,\dot{\rho} = \rho''\left(\frac{J_z^2}{m^2}\,\rho^4\right) + 2\,\rho'^2\,\frac{J_z^2}{m^2}\,\rho^3\end{aligned} \tag{3.10}$$

and putting this into the equation of motion gives

$$0 = \frac{J_z^2}{m}\,\rho'' + \frac{2\,J_z^2}{m}\,\rho'^2\rho^{-1} - 2\,m\left(\rho'^2\,\frac{J_z^2}{m^2}\right)\,\rho^{-1} + \frac{J_z^2\,\rho}{m} + \frac{dU}{d\rho} \tag{3.11}$$

Or, finally,

$$\boxed{-\frac{dU}{d\rho} = \frac{J_z^2}{m}\,\rho'' + \frac{J_z^2}{m}\,\rho.} \tag{3.12}$$

Where's the gravity? In the potential, the potential for Newtonian gravity is $U = -\frac{G\,m\,M}{r} = -G\,m\,M\,\rho$, and that's easy to differentiate:

$$G\,M\left(\frac{m}{J_z}\right)^2 = \rho'' + \rho \rightarrow \rho(\phi) = G\,M\left(\frac{m}{J_z}\right)^2 + \alpha\,\cos\phi + \beta\,\sin\phi. \quad (3.13)$$

Simple, right? What type of solution is this? Keep in mind that $G\,M\,\frac{m^2}{J_z^2}$ is just a constant, so what we really have is:

$$r(\phi) = \frac{1}{A + \alpha\,\cos\phi + \beta\,\sin\phi}. \quad (3.14)$$

As a harmonic oscillator in ϕ, let's agree to start at "velocity zero" at $\phi = 0$. This means that

$$r'(\phi) = -(A + \alpha\,\cos\phi + \beta\,\sin\phi)^{-2}(-\alpha\,\sin\phi + \beta\,\cos\phi)\,|_{\phi=0} = 0 \rightarrow \beta = 0 \quad (3.15)$$

so that

$$\boxed{r(\phi) = \frac{1}{A + \alpha\,\cos\phi},} \quad (3.16)$$

and this familiar solution is shown in Figure 3.1.

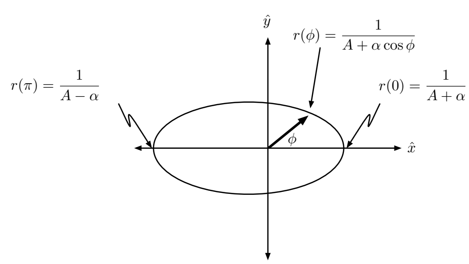

Figure 3.1: Ellipse in $r(\phi)$ parametrization

That's the story with elliptical orbits. We used the Lagrange approach to find a first integral of the motion (J_z), then we solved the problem using ϕ as the parameter for the curve $(r(\phi), \phi)$). There are a couple of things we will be dropping from here on out – the first is to set $G = 1$, this just changes how we measure masses. We can also set the test mass $m = 1$, it cannot be involved in the motion – this choice just rescales J_z.

3.2 More on Tensors

In preparation for the Hamiltonian form, I want to be more careful with my definition of tensors. Again, we will be covering this in a lot of detail relatively soon, but I want to discuss a few key points right now.

There are two varieties of tensor with which we will need to become familiar. A tensor, for us, is defined by its behavior under a coordinate transformation. In particular, if we have an indexed object $f^\alpha(x)$ that depends on a set of coordinates x and we change to a new coordinate system $x \rightarrow x'$, then a *contravariant tensor* transforms via:

$$\boxed{f'^\alpha(x') = f^\nu(x')\,\frac{\partial x'^\alpha}{\partial x^\nu},} \tag{3.17}$$

and it is important to remember that the right-hand-side should (morally speaking) be expressed entirely in terms of the new coordinate system (so that, for example, what we mean by $f^\nu(x')$ is really $f^\nu(x(x'))$, i.e. we take the original vector in x coordinates, and rewrite the x in terms of the x', the same holds for the transformation factor $\frac{\partial x'^\alpha}{\partial x^\nu}$).

A *covariant* tensor responds to coordinate transformations according to

$$\boxed{f'_\alpha(x') = f_\nu\,\frac{\partial x^\nu}{\partial x'^\alpha}.} \tag{3.18}$$

The standard example of a contravariant tensor is the coordinate differential dx^α, and the most famous covariant tensor is the gradient: $\frac{\partial \phi}{\partial x^\alpha} \equiv \phi_{,\alpha}$.

The fun continues with more indices – we introduce the appropriate transformation factor for each index:

$$\begin{aligned} f'^{\alpha\beta}(x') &= \frac{\partial x'^\alpha}{\partial x^\mu}\,\frac{\partial x'^\beta}{\partial x^\nu}\,f^{\mu\nu} \\ f'_{\alpha\beta}(x') &= \frac{\partial x^\mu}{\partial x'^\alpha}\,\frac{\partial x^\nu}{\partial x'^\beta}\,f_{\mu\nu} \end{aligned} \tag{3.19}$$

There is no particular reason to imagine that these two different objects (covariant and contravariant tensors) are connected, but within the confines of the geometry we will be studying, they are. We use precisely the metric and its matrix-inverse to define raising and lowering operations. So, for example, we have

$$f_\alpha = g_{\alpha\nu}\,f^\nu \tag{3.20}$$

and if we set $g^{\alpha\nu} \equiv (g_{\mu\nu})^{-1}$ the matrix inverse of $g_{\mu\nu}$, then

$$f^\alpha = g^{\alpha\nu} f_\nu. \tag{3.21}$$

The importance of the inverse is to recover:

$$f_\alpha = g_{\alpha\nu} f^\nu = g_{\alpha\nu} \left(g^{\nu\beta} f_\beta \right) = \delta^\beta_\alpha f_\beta = f_\alpha. \tag{3.22}$$

The coordinate definitions you know and love are $dx^\mu \doteq (dx, dy, dz)$, and this begs the question – how has the distinction between up and down never come up in the past? We're used to writing $\mathbf{x}$, that doesn't appear to be either up or down.

The answer can be made clear with a simple example. Suppose we take non-orthogonal axes in two-dimensions. Then we can measure, in these coordinates, the projection (parallel to the skewed axes) of the vector shown in Figure 3.2. We would write $v^\alpha = \begin{pmatrix} a \\ b \end{pmatrix}$.

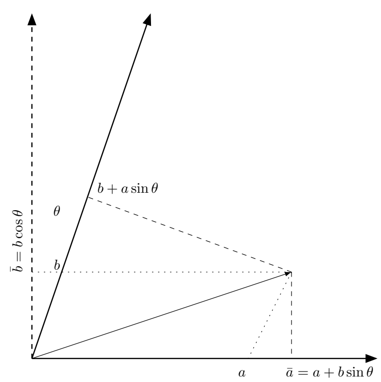

Figure 3.2: The difference between covariant and contravariant.

We want to find the metric in the skewed coordinate system – so let's calculate the Cartesian components of the vector, they are:

$$\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} 1 & \sin\theta \\ 0 & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{3.23}$$

and then in this notation, the length of the vector is:

$$\begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.24}$$

By definition, the matrix in the middle is the metric for the non-orthogonal space. Now if we take the vector $v^\alpha \doteq \begin{pmatrix} a \\ b \end{pmatrix}$, then the lowered form is $v_\alpha = v^\beta g_{\beta\alpha}$, and performing the contraction, we have

$$v_\alpha \doteq \begin{pmatrix} a + b \sin\theta \\ b + a \sin\theta \end{pmatrix}. \tag{3.25}$$

Referring to the figure again, these two quantities are indicated by the dashed lines, i.e. the components of the covariant vector v_α are the coordinates w.r.t. the perpendicular projection onto the non-orthogonal basis.

If we always use orthonormal axes, the distinction never arises, since in that case, the perpendicular projection is identical to the parallel projection.