Bending of Light in Schwarzschild Geometry.

Lecture 31

Physics 411
Classical Mechanics II

November 12th, 2007

Today, we look at an entirely new type of geodesic for the Schwarzschild geometry – light. In classical, Newtonian gravity, this is not something we can sensibly ask about – but as a geodesic on a manifold, light is not that strange – we ask, for the physical particulars of the photon (null tangent vectors) what a "straight line" looks like, as always. There is some confusion about the physical interpretation of the "proper time" – that is naturally built in to the discussion and had better exist. After all, the proper time of light is an ill-defined concept.

It is worth noting the null-vector ($ds^2 = 0$) nature of these photon trajectories, they crop up again when we talk about coordinate transformations and one can draw an interesting parallel between the coordinate time and the "proper time".

31.1 Light in Schwarzschild Geometry

Light is "just another" test particle in GR, on a semi-equal footing with (time-like) particles. It does make some sense to separate the treatments, but as you will see, there is almost no difference in the calculation.

How do we describe light? Well, its rest mass is zero, i.e. the invariant four-momentum magnitude is (in Minkowski space-time) $-E^2 + \vec{p} \cdot \vec{p} = 0$. That tells us in our Lagrange setting, that $L = H = 0$ rather than the $-\frac{1}{2}$ we had in the previous section. On the metric side, this is just another option for us – for a metric with a time-component (typified by the -1 in η_{00}) we cannot say that a "length" of zero implies that the vector is zero. Material particles must have "time-like" or $d\tau^2 = -1$ character (for affine parametrization with τ the proper time), then "light-like" means $d\tau^2 = 0$

and "space-like" has $d\tau^2 = 1$, the normalization of spatial coordinate axes. Regardless, here we will deal with light-like "particles": light. Notice that since $d\tau^2 = 0$, our notion of proper time for light is no longer sensible, we can still parametrize the curve, though, and without confusion, let me call this parametrization τ.

The setup is identical to the Test Particle section, so we can cut to the chase – start with the Schwarzschild metric written in the ρ coordinate, we again set the motion in the $\theta = \frac{\pi}{2}$ plane, identify J_z as the canonical momentum associated with ϕ (and so constant), and $-E$, the canonical momentum for t, also constant. The first difference occurs at the "$L = -\frac{1}{2}$" of massive particle analysis. For light, with $L = 0$ we have:

$$\begin{aligned} 0 = L &= \frac{1}{2}\left(J_z^2\,\rho^2 - \frac{E^2\,\rho^4 - \dot{\rho}^2}{\rho^4\,(1 - 2\,M\,\rho)}\right) \\ &\downarrow \\ \dot{\rho}^2 &= \rho^4\left(\frac{E^2}{J_z^2} - \rho^2 + 2\,M\,\rho^3\right). \end{aligned} \tag{31.1}$$

Switching again to ϕ derivatives via $\dot{\rho} = \rho'\,J_z\,\rho^2$ gives us the analogue of $\rho'' = -\rho + \frac{M}{J_z^2} + 3\,M\,\rho^2$:

$$\rho'' = -\rho + 3\,M\,\rho^2. \tag{31.2}$$

Now let's think a bit about the meaning of the $M = 0$ solution, the one we would get from, for example, special relativity. It says that the trajectory of light, written in parametric form in the $x - y$ plane is

$$\begin{aligned} x &= r\,\cos\phi = \frac{\cos\phi}{\alpha\,\cos\phi + \beta\,\sin\phi} \\ y &= r\,\sin\phi = \frac{\sin\phi}{\alpha\,\cos\phi + \beta\,\sin\phi}, \end{aligned} \tag{31.3}$$

and we can set the coordinates such that in the $M = 0$ limit, we have a horizontal line passing a distance R from the origin by putting $\alpha = 0$, $\beta = R^{-1}$ (never mind the funny $\cot\phi$, with these choices $y = R$ always, clearly a line – x goes from $-\infty \longrightarrow \infty$). So we would say that light passes unperturbed through empty space (or in a space-time that ignores GR, like Minkowski).

Again, the $M \neq 0$ case is best handled through perturbation – let us take a solution of the form $\rho = \frac{\sin\phi}{R} + \epsilon\,\tilde{\rho}$, then we see that

$$-\frac{1}{R}\,\sin\phi + \epsilon\,\tilde{\rho}'' = -\left(\frac{1}{R}\,\sin\phi + \epsilon\,\tilde{\rho}\right) + 3\,M\left(\frac{1}{R}\,\sin\phi + \epsilon\,\tilde{\rho}\right)^2 \tag{31.4}$$

and if we associate our ϵ expansion parameter with the mass M of the gravity-producing object (this is the point here: for $M = 0$, we get a line) then to first order, we can read off the equation for $\tilde{\rho}$:

$$\tilde{\rho}'' = -\tilde{\rho} + \frac{3}{R^2} \sin^2 \phi. \tag{31.5}$$

31.1.1 Aside: Generic Solution of Inhomogenous, Linear ODEs

I just want to take a moment to talk about the simple relationship between source-free solutions of first order ODEs and their driven solutions – take:

$$f'(x) = A\, f(x) \tag{31.6}$$

where we are (here and only here) viewing $f(x)$ as a vector and A as a matrix (x is just the single parameter in the problem, time for example). The solution is:

$$f(x) = e^{A\,x}\, f(0) \tag{31.7}$$

where we mean matrix exponentiation (defined in terms of the powers of matrices just as with normal exponentials).

Suppose we add a source $G(x)$, some function of x, then we have

$$f'(x) = A\, f(x) + G(x), \tag{31.8}$$

but that looks a lot like a change of center – as if we added a constant to a spring equation for example. Motivated by this, let's see what happens if we add a continuous shift, consider:

$$\bar{f}(x) \equiv e^{A\,x}\,(f(0) + g(x)) \rightarrow \bar{f}'(x) = A\,\bar{f} + e^{A\,x}\, g'(x). \tag{31.9}$$

That's perfect, compare with the above, and we see that $\bar{f}(x)$ solves (31.8) if we set

$$e^{A\,x}\, g'(x) = G(x) \rightarrow g'(x) = e^{-A\,x}\, G(x) \rightarrow g(x) = \int_0^x e^{-A\,\bar{x}}\, G(\bar{x})\, d\bar{x} \tag{31.10}$$

where we have made some assumptions effectively about the invertibility of A, but don't worry for now. Going back, the form for $\bar{f}(x)$ is

$$\bar{f}(x) = e^{A\,x}\, f(0) + e^{A\,x} \int_0^x e^{-A\,\bar{x}}\, G(\bar{x})\, d\bar{x}. \tag{31.11}$$

Solving inhomogenous ODEs amounts to exponentiating a matrix and doing some integration. Let's apply this idea to the above problem, the homogenous part is given by

$$\frac{d}{d\phi} \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}' \end{pmatrix}, \tag{31.12}$$

and we can immediately tell what the matrix A in the above has as its exponential – we know the solution is $\rho(\phi) = \rho(0)\, \cos\phi + \dot{\rho}(0)\, \sin\phi$ so

$$e^{A\,\phi} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \rightarrow e^{-A\,\phi} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{31.13}$$

With this "fundamental" solution, let's calculate the source integral:

$$\begin{aligned} e^{A\,\phi} \int_0^\phi e^{-A\,\bar{\phi}} \begin{pmatrix} 0 \\ \frac{\sin^2\bar{\phi}}{R^2} \end{pmatrix} d\bar{\phi} &= \frac{1}{R^2}\, e^{A\,\phi} \int_0^\phi \begin{pmatrix} -\sin^3\bar{\phi} \\ \cos\bar{\phi}\, \sin^2\bar{\phi} \end{pmatrix} d\bar{\phi} \\ &= \left(\frac{1}{3\,R^2}\right) \begin{pmatrix} 4\, \sin^4(\frac{1}{2}\,\phi) \\ 2\, \sin\phi - \sin(2\,\phi) \end{pmatrix} \end{aligned} \tag{31.14}$$

All right, you could have done it faster, but this is generic, isn't that great?! Anyway, the point of all this:

$$\begin{aligned} \tilde{\rho}(\phi) &= \alpha\, \cos\phi + \beta\, \sin\phi + \frac{\sin^4(\frac{1}{2}\,\phi)}{R^2} \\ &= \alpha\, \cos\phi + \beta\, \sin\phi + \frac{3 - 4\, \cos\phi + \cos(2\,\phi)}{2\,R^2} \end{aligned} \tag{31.15}$$

and we can get rid of the first two terms – this is all within the perturbation, and we already have the proper normalization for straight line motion, we can also drop the $4\cos\phi$ term by appropriate choice of α (in order to get rid of both of them).

The punch line of the above is that

$$\rho(\phi) = \frac{\sin\phi}{R} + M\,\frac{(1+\cos^2\phi)}{R^2} \tag{31.16}$$

solves (31.2) to first order in $\epsilon \sim M$. What do these trajectories look like? They are bent about the $\phi = \frac{1}{2}\pi$ axis – reaching a closest approach to the massive body at $r_{graze} = \frac{R^2}{M+R}$ (setting $\phi = \frac{1}{2}\pi$) and crossing the body's plane at $\phi = 0 \rightarrow r_{plane} = \frac{R^2}{2\,M}$. Now part of our assumption of small M is that the bending is slight, we can see this at the axis crossing, $r_{plane} \sim M^{-1}$, so if we are sitting far away (at infinity effectively), the angle made w.r.t. the axis is small as shown in Figure 31.1, this is our viewing angle ϕ_v.

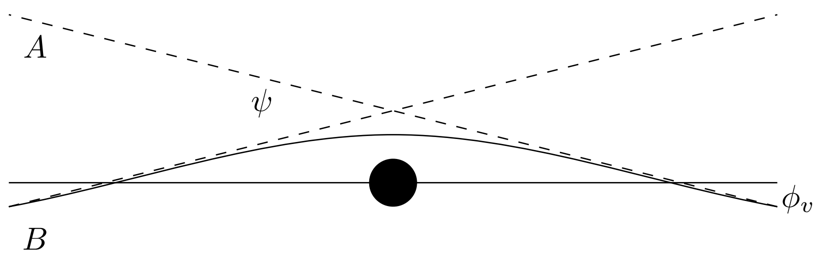

Figure 31.1: The path of light, deflected about a massive object. We are viewing from far away with a viewing angle ϕ_v, the light is "deflected" by the angle ψ.

At spatial infinity, $\rho = 0$, and we assume this is where our viewing platform is set up. Then for $\phi = -\phi_v$ (ϕ in this picture starts at zero on the axis, so we are at a negative value for the angle) and ϕ_v small, we have

$$0 = \rho(\phi_v) \approx \frac{2\,M}{R^2} - \frac{\phi_v}{R} \rightarrow \phi_v = \frac{2\,M}{R} \tag{31.17}$$

and by symmetry, here, we have the deflection angle $\psi = 2\,\phi_v$ so

$$\boxed{\psi = \frac{4\,M}{R}} \tag{31.18}$$

using $R = R_{sun} = 6.96 \times 10^5$ km and $M = 1.483$ km, the mass of the sun in kilometers,

$$\begin{aligned}\psi &\approx 8.523 \times 10^{-6} = 4.8 \times 10^{-4}\text{ deg} \times \frac{60^2\text{arcsec}}{1\text{deg}} \\ &\approx 1.76\text{ arcsec.}\end{aligned} \tag{31.19}$$

In other words, if we assume we are looking along the straight dotted line of our viewing angle in Figure 31.1, we would say that the light we see comes from point A, while due to bending it came from point B, a deflection of ψ. In the case of grazing deflection (that's why we set $R = R_{sun}$) this is still an incredibly small effect.