Hamiltonian Solution II

Lecture 6

Physics 411
Classical Mechanics II

September 10th 2007

Using the Killing vectors we developed last time, and associated constants, I will finish with the solution of orbital motion in the Hamiltonian framework.

The lesson of the Hamiltonian equations of motions, together with the constants we can get from the Killing vectors is: if possible, never form them. In more complicated settings, of course, one must write them down, but by the time you've done that, it's almost always a computational (meaning "with a real computer") effort to solve them. So the solution procedure here looks quite a bit different than the equivalent Lagrange approach.

As I've mentioned before, and will probably point out a few more times: one of the interesting things about GR as a topic is its lack of forces, which makes $L = H$, and we can blend techniques between the two points of view.

6.1 Interpreting the Killing Vector

Remember where we were: we had solved the PDE that comes from Killing's equation:

$$p^\alpha\, p^\beta \left(\frac{1}{2}\, g_{\alpha\beta,\gamma}\, f^\gamma + g_{\alpha\gamma}\, f^\gamma{}_{,\beta}\right) = 0 \leftrightarrow \boxed{f_{(\alpha;\beta)} = 0} \tag{6.1}$$

We made a vector ansatz for f^α, most importantly, we took away the f^r component. The motivation there came from the second piece of the $[H, J] = 0$ requirement:

$$U_{,\alpha}\, f^\alpha = 0 \rightarrow \frac{\partial U}{\partial r}\, f^r = 0 \tag{6.2}$$

and for central potentials, this must have $f^r = 0$ (since the derivative is the force). Again, on the relativistic side, this is not an issue, and Killing vectors come along for the ride once a metric is specified.

With the ansatz in place, we solved the PDE to get a vector of the form

$$f^\alpha \doteq \begin{pmatrix} 0 \\ A\,\cos\phi + B\,\sin\phi \\ F + \cot\theta\,(B\,\cos\phi - A\,\sin\phi) \end{pmatrix}. \tag{6.3}$$

Now the question is – what is this? Well, the first question is, how *many* is this? We have three constants of integration floating around. These actually correspond to three separate Killing vectors,

$$f_1^\alpha \doteq \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix} \quad f_2^\alpha \doteq \begin{pmatrix} 0 \\ A\,\cos\phi \\ -A\,\cot\theta\sin\phi \end{pmatrix} \quad f_3^\alpha \doteq \begin{pmatrix} 0 \\ B\,\sin\phi \\ B\,\cot\theta\cos\phi \end{pmatrix}. \tag{6.4}$$

Each of these vectors is involved in a transformation, $X^\alpha = x^\alpha + \epsilon\, f^\alpha$, so the second vector, for example, induces:

$$R = r \quad \Theta = \theta + \epsilon\,(A\,\cos\phi) \quad \Phi = \phi + \epsilon\,(-A\,\cot\theta\sin\phi). \tag{6.5}$$

This is an infinitesimal transformation, it is also, by virtue of its derivation, canonical.

What are the conserved quantities? For each of these transformations, we should have a constant of the motion – precisely the generator J. Remember the form of J:

$$J = p_\alpha\, f^\alpha \tag{6.6}$$

so suppose we take the first of the three transformations, then

$$J = p_\phi\, F. \tag{6.7}$$

The F is really just a normalization, taken care of by ϵ – here we learn that $J = p_\phi$ is conserved and this comes from the transformation $\Phi = \phi + \epsilon$. That's something we already knew, but gives a clue about the rest of the Killing vectors – angular momentum conservation comes from the rotational invariance of the equations of motion. If we took the Cartesian transformation corresponding to infinitesimal rotation about a vector $\mathbf{\Omega} \equiv (\omega_x, \omega_y, \omega_z)$ [1] ,

$$\boxed{\mathbf{X} = \mathbf{x} + \epsilon\,\mathbf{\Omega}\,\times\,\mathbf{x}} \tag{6.8}$$

[1] To see (6.8), consider $\mathbf{\Omega} = \phi\,\hat{z}$, and set a generic vector $\mathbf{v}$ in the $x - z$ plane. Then from Figure 6.1, we rotate about the z-axis, expressible as $\mathbf{v}' = \mathbf{v} + \mathbf{\Omega}\,\times\,\mathbf{v}$.

then, upon transformation to spherical coordinates, we would find (to first order in ϵ):

$$\begin{aligned}\Theta &= \theta + \epsilon\,(\omega_y\,\cos\phi - \omega_x\,\sin\phi)\\ \Phi &= \phi + \epsilon\,(\omega_z - \cot\theta\,(\omega_x\,\cos\phi + \omega_y\,\sin\phi)).\end{aligned} \tag{6.9}$$

Evidently, for our choice of constants (A, B, F), we are describing the vector $\mathbf{\Omega} \doteq (-B, A, F)$.

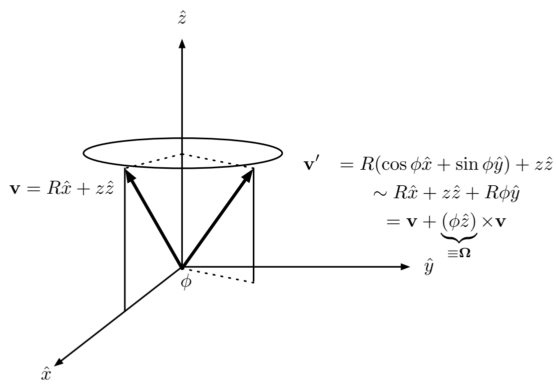

Figure 6.1: An infinitesimal rotation about the $\mathbf{\Omega} \sim \hat{z}$ axis for an arbitrary vector $\mathbf{v}$.

consider the generators J that comes from the other two vectors:

$$\boxed{\begin{aligned}J_x &= p_\theta\,\sin\phi + p_\phi\,\cot\theta\,\cos\phi\\ J_y &= p_\theta\,\cos\phi - p_\phi\,\cot\theta\,\sin\phi\\ J_z &= p_\phi\end{aligned}} \tag{6.10}$$

this gives us the full complement of angular momenta, agreeing with the spherical form for $\mathbf{x} \times \mathbf{p}$.

So what?

6.2 Solution of Hamiltonian System

Let's count: we have three constants of the motion so far, these can be used to eliminate momenta or coordinates. In addition to these three, we have H itself – after all, it's hard to imagine a zero more compelling than $[H, H]$, so that's four constants of the motion.

We are not free to do anything we like, there is no p_r in the angular momentum constants, for example. Looking specifically at those, setting (J_x, J_y, J_z) must lead to fixing two components of momentum and θ, if we take the J_z equation at face value

$$\begin{aligned} J_x &= p_\theta \sin\phi + J_z \cot\theta \cos\phi \\ J_y &= p_\theta \cos\phi - J_z \cot\theta \sin\phi \end{aligned} \tag{6.11}$$

then suppose we want $J_x = J_y = 0$, we get two equations for p_θ

$$p_\theta = -J_z \cot\theta \cot\phi \qquad p_\theta = J_z \cot\theta \tan\phi \tag{6.12}$$

and this can only be zero for $\cot\theta = 0 \to \theta = \pm\frac{1}{2}\pi$.

This is the source of the difficulty we had on the Lagrange side. By setting the motion in the equatorial plane, we were effectively setting $J_x = J_y = 0$, leaving us with J_z. We do not have another choice of plane with $J_x = J_y = 0$.

With that in place, though, we automatically get $p_\theta = 0$ along with $\theta = \frac{1}{2}\pi$, and we can use this in the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2} p_\alpha g^{\alpha\beta} p_\beta + U(r) \\ &= \frac{1}{2}\left(p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2\theta} p_\phi^2\right) + U(r) \\ &= \frac{1}{2}\left(p_r^2 + \frac{1}{r^2} J_z^2\right) + U(r) \end{aligned} \tag{6.13}$$

Now we can use the constancy of H to solve for p_r^2 – let's call $H = E$, the numerical constant, then solving for the p_r component of momentum gives

$$p_r^2 = 2E - 2U(r) - \frac{1}{r^2} J_z^2. \tag{6.14}$$

To connect this to the r velocity, we can use the equation of motion (or definition of canonical momentum from the Lagrangian) – we have $p_r = \dot{r}$, and we can specialize to the Newtonian gravitational potential to get:

$$(\dot{r})^2 = 2E + 2\frac{M}{r} - \frac{J_z^2}{r^2} = \frac{1}{r^2}\left(2Er^2 + 2Mr - J_z^2\right). \tag{6.15}$$

It is difficult to imagine solving this if we took the square root, there is a sign issue there, of course, but also the ODE gets more complicated. Let's think about what we can get out of the above with no work. The first thing we notice is that the term in parenthesis is quadratic in r. The roots of this quadratic are points where $\dot{r} = 0$, which we associate with turning points of the motion – suppose we explicitly factor the quadratic into its roots

$$\begin{aligned} \dot{r}^2 &= \frac{2\,E}{r^2}(r - r_+)\,(r - r_-) \\ r_+ &= \frac{1}{2}\left(-\frac{M}{E} + \sqrt{\left(\frac{M}{E}\right)^2 + 2\,\frac{J_z^2}{E}}\right) \\ r_- &= \frac{1}{2}\left(-\frac{M}{E} - \sqrt{\left(\frac{M}{E}\right)^2 + 2\,\frac{J_z^2}{E}}\right). \end{aligned} \tag{6.16}$$

In a situation like this, a change of variables is in order – we know that whatever r is, it has two turning points, with r_- further away than r_+, so we are tempted to set

$$r(\psi) = \frac{p}{1 + e\,\cos\psi} \tag{6.17}$$

this is, not surprisingly, an ellipse – the two turning points of an ellipse are

$$r(0) \equiv r_p = \frac{p}{1+e} \qquad r(\pi) \equiv r_a = \frac{p}{1-e} \tag{6.18}$$

and we can associate these with the turning points of our $\dot{r}$ equation – this tells us that

$$\frac{p}{1+e} = r_+ \qquad \frac{p}{1-e} = r_- \longrightarrow \boxed{J_z = \sqrt{M\,p} \qquad E = \frac{M\,(e^2-1)}{2\,p}} \tag{6.19}$$

Meanwhile, we also have

$$\dot{r} = \frac{p\,e\,\sin\psi\,\dot{\psi}}{(1 + e\,\cos\psi)^2} \tag{6.20}$$

So our quadratic equation reads

$$\begin{aligned} (\dot{r})^2 = \frac{p^2\,e^2\,\sin^2\psi\,\dot{\psi}^2}{(1 + e\,\cos\psi)^4} &= 2\,E\frac{(1 + e\,\cos\phi)^2}{p^2}\left(\frac{p}{1 + e\,\cos\psi} - \frac{p}{1+e}\right)\left(\frac{p}{1 + e\,\cos\psi} - \frac{p}{1-e}\right) \\ &= 2\,\frac{M\,(e^2-1)}{2\,p}\,\frac{(p + e\,\cos\psi)^2}{p^2}\left(\frac{e\,(p - p\,\cos\psi)\,(-e(1 + p\,\cos\psi))}{(1 + e\,\cos\psi)^2\,(1 - e^2)}\right) \\ &= -\frac{M}{p}\,(-e^2)\left(\sin^2\psi\right) \end{aligned} \tag{6.21}$$

which gives

$$(\dot{\psi})^2 = \frac{M\,(1 + e\,\cos\psi)^4}{p^3}. \tag{6.22}$$

This is a particularly nice parametrization, and the starting point for many computational investigations when the geometry becomes more complicated.

But what is the relationship of ψ to, for example, ϕ? We'll perform a change-of-variables:

$$\begin{aligned}\dot{\psi} &= \frac{d\psi}{d\phi}\,\frac{d\phi}{dt} = \psi'\,\dot{\phi} = \psi'\left(\frac{J_z}{r^2}\right)\\ &= \psi'\,\sqrt{p\,M}\left(\frac{1 + e\,\cos\psi}{p}\right)^2\end{aligned} \tag{6.23}$$

and then

$$(\dot{\psi})^2 = \psi'^2\,\frac{M}{p^3}\,(1 + e\,\cos\psi)^4 = \frac{M}{p^3}\,(1 + e\,\cos\psi)^4 \tag{6.24}$$

conclusion: $\psi' = 1$, or $\psi = \phi$ (plus an arbitrary phase that we aren't interested in).

So we discover,

$$r(\psi) = r(\phi) = \frac{p}{1 + e\,\cos\phi} \tag{6.25}$$

as before, with (p, e) related to the energy and angular momentum of the orbit through

$$J_z = \sqrt{M\,p} \qquad E = \frac{M\,(e^2 - 1)}{2\,p}. \tag{6.26}$$

Incidentally, (p, e) are intrinsically greater than zero, but bound orbits have $E < 0$ (they're caught in a well), so for bound orbits, we want $e^2 < 1$. J_z can be positive or negative, $J_z = \pm\sqrt{M\,p}$ corresponding to the direction (counterclockwise or clockwise) of the orbit.

We are, finally, done with orbital motion. I hope that I have convinced you that L and H lead to different ways of viewing the solutions, but they are entirely equivalent (as is evidenced by the fact that the solutions are identical!).

It is not uncommon to characterize the more involved space-times (i.e. $g_{\mu\nu}$) that arise in general relativity in terms of the particle orbits they produce. This somewhat long introduction will, I hope, prove useful when you are handed some non-obvious metric and forced to discuss it in a reasonable way.

6.3 Temporal Evolution

Starting from (6.14), we can analyze the motion of any polynomial central potential we wish using the techniques discussed above. The move to $\psi = \phi$ parametrization is a particularly useful choice, and can tell us a lot about the motion of the body, but we lose all temporal evolution information. We can recover the time dependence *a posteriori* by explicit differentiation – for our gravitational example:

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} \tag{6.27}$$

and we know $\dot{\phi} = \frac{J_z}{r^2}$. We can write a small change in t in terms of a small change in ϕ from the above:

$$dt = \frac{dr}{\frac{dr}{d\phi}\,\dot{\phi}} = \frac{d\phi}{\dot{\phi}} = \frac{p^2\,d\phi}{J_z\,(1 + e\,\cos\phi)^2}, \tag{6.28}$$

and we can integrate this from $\phi : 0 \longrightarrow 2\pi$ to find the total orbital period:

$$T = \int_0^{2\pi} \frac{p^2\,d\phi}{J_z\,(1 + e\,\cos\phi)^2} = \frac{J_z^3}{M^2}\,\frac{2\,\pi\,i}{(e^2 - 1)^{3/2}} \tag{6.29}$$

where we have used $p^2 = \frac{J_z^4}{M^2}$. Note that the semi-major axis of the orbit is just

$$a \equiv \frac{p}{1+e} + \frac{p}{1-e} = \frac{p}{1-e^2} \tag{6.30}$$

so the period can be written in terms of this geometric parameter:

$$T = -2\,\sqrt{M}\,\pi\,a^{3/2}, \tag{6.31}$$

ignoring the minus sign, which is one of Kepler's laws.

The fact that equal areas are swept out in equal times (another of Kepler's observations) comes directly from the infinitesimal area of an ellipse and angular momentum conservation.

For the small triangle shown (enlarged) in Figure 6.2, we have approximate area $dA = \frac{1}{2}\,r^2\,d\phi$, and if this is swept out in time dt, we can write:

$$\frac{dA}{dt} = \frac{1}{2}\,r^2\,\dot{\phi} = \frac{1}{2}\,J_z, \tag{6.32}$$

a constant.

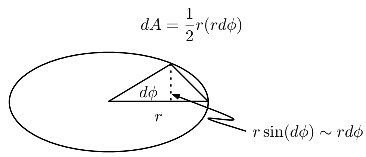

Figure 6.2: An ellipse in $r(\phi)$ parametrization – we make a small triangle and use "one half base times height" with appropriate approximations to find dA, the infinitesimal area.

6.4 Reparametrization

We chose, in this case, to reparametrize the Hamiltonian itself. Since it was a constant, it furnished a first-order differential equation, and we used the chain rule to rewrite $\dot{r} = \frac{dr}{d\phi}\dot{\phi}$ – our expression for $\dot{\phi}$ in terms of r facilitated this move. We can also reparametrize in the equations of motion, either Hamiltonian or Lagrangian. It is interesting to ask, then, if it is possible to directly reparametrize a Lagrangian. It is, of course, but the procedure makes heavy use of the action $S = \int L\,dt$ – simply reparametrizing L is not directly accessible since L is not a constant of the motion, in general, and its physics is dictated by its equations of motion. This is an important issue, and one to which we will return during our discussion of relativistic Lagrangians and Hamiltonians, where reparametrization (and invariance) play an even more central role.