Problem Set 1

Classical Mechanics II
Physics 411

Due on September 7th 2007

Problem 1.1

For a matrix $\mathbb{A}$ with entries A^{ij} [1] and a vector $\mathbf{x}$ with entries x_k, write the matrix-vector products $\mathbb{A}\,\mathbf{x}$ and $\mathbf{x}^T\mathbb{A}$ in indexed notation.

Problem 1.2

Start with the "usual" Cartesian Lagrangian for a central potential,

$$L = \frac{1}{2}\,m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(\sqrt{x^2 + y^2 + z^2}\right) \tag{2}$$

and transform the coordinates (and velocities) directly (no metric allowed) to find the Lagrangian associated with a central potential in cylindrical coordinates, with $x^1 = s$, $x^2 = \phi$, $x^3 = z$. From the Lagrangian itself (most notably, its kinetic term), write the metric associated with cylindrical coordinates.

Problem 1.3

a. Given a second rank tensor $T_{\mu\nu}$, often viewed as an $N \times N$ matrix (for a space of dimension N), show by explicit construction that one can always decompose $T_{\mu\nu}$ into a symmetric ($S_{\mu\nu} = S_{\nu\mu}$) and antisymmetric part ($A_{\mu\nu} = -A_{\nu\mu}$) via $T_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu}$. The symmetric portion is often denoted $T_{(\mu\nu)}$, and the antisymmetric $T_{[\mu\nu]}$.

[1] So that in two dimensions, for example,

$$\mathbb{A} \doteq \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}. \tag{1}$$

b. In terms of the decomposition, $T_{\mu\nu} = S_{\mu\nu} + A_{\mu\nu}$, evaluate the sums

$$T_{\mu\nu}\, Q^{\mu\nu} \qquad T_{\mu\nu}\, P^{\mu\nu} \tag{3}$$

with $Q^{\mu\nu}$ symmetric, and $P^{\mu\nu}$ antisymmetric.

c. Using the decomposition above, show that, as stated in class:

$$\dot{x}^\nu\, \dot{x}^\gamma \left(\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) = \frac{1}{2}\, \dot{x}^\nu\, \dot{x}^\gamma \left(\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) \tag{4}$$

(note that $\dot{x}^\nu\, \dot{x}^\gamma$ plays the role of a symmetric second rank tensor).

Problem 1.4

a. For the metric $g_{\mu\nu}$ in spherical coordinates, and coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, find the r component of the equation of motion (i.e. $\alpha = 1$)

$$m g_{\alpha\nu} \ddot{x}^\nu + m \dot{x}^\nu \dot{x}^\gamma \frac{1}{2} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial g_{\gamma\nu}}{\partial x^\alpha} \right) = -\frac{\partial U}{\partial x^\alpha}. \tag{5}$$

b. Starting from the Lagrangian in spherical coordinates, calculate the r equation of motion directly from

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \tag{6}$$

and verify that you get the same result.