Problem Set 2

Classical Mechanics II
Physics 411

Due on September 14th 2007

Problem 2.1

We have been focusing on bound orbits, but one can also approach a massive object directly along a line, this is radial infall.

a. From our radial equation for the ϕ-parametrized $\rho(\phi) = 1/r(\phi)$ curve, we had, for arbitrary $U(\rho)$:

$$\frac{J_z^2}{m}\big(\rho''(\phi) + \rho(\phi)\big) = -\frac{dU(\rho)}{d\rho}, \tag{1}$$

can this equation be used to develop the ODE appropriate for radial infall with the Newtonian point potential? (i.e. A particle falls inward from $r(t = 0) = R$ with $\dot{r}(t = -\infty) = 0$ towards a spherically symmetric central body with mass M sitting at $r = 0$.) If not, explain why, if so, prepare to solve the relevant ODE for $r(t)$ in the next part.

b. Solve the radial infall problem with initial conditions from part a. – i.e. find $r(t)$ appropriate for a particle of mass m falling *towards* $r = 0$ along a straight line – assume a spherically symmetric massive body is located at $r = 0$ with mass M.

Problem 2.2

Inspired by conservation of energy $H = E$ (Hamiltonian is a constant), and the Lagrangian for orbital motion in the $\theta = \frac{1}{2}\pi$ plane:

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r) \tag{2}$$

with $U(r) = -\frac{M}{r}$ as usual, one can reinterpret orbital motion as one-dimensional motion in r (the Hamiltonian involves only the r coordinate and the constant associated with angular momentum, J_z).

a. Write the effective potential energy (i.e. the potential energy for the one-dimensional problem) in this setting. Make a sketch of the effective potential, label its behavior for $r \sim 0$, $r \sim \infty$. On your sketch, identify any zero-crossings and minima (both location r and value U_{min}).

b. Solve for $r(t)$ when $E = U_{min}$ from part a. What does the full two-dimensional solution look like in the $x - y$ plane for this case?

Problem 2.3

a. For an ellipse parametrized by:

$$r(\phi) = \frac{p}{1 + e\cos\phi} \tag{3}$$

sketch the trajectory (in the (x, y) plane) for $(p, e) = \left(1, \frac{1}{2}\right)$, $(p, e) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $(p, e) = \left(1, \frac{1}{4}\right)$ by considering the points defined by $\phi = \left\{0, \frac{\pi}{2}, \pi\right\}$. What values of (p, e) correspond to a circle of radius R?

b. Write (r_a, r_p) (the radii for aphelion – furthest and perihelion – closest approach to the central body) in terms of (p, e).

c. Using the above, find the relationship between the constants of integration (α, J_z) in

$$r(\phi) = \frac{1}{\frac{M}{J_z^2} + \alpha\cos\phi} \tag{4}$$

(from our solution using the Lagrangian) and (r_a, r_p) – that is, find $\alpha(r_a, r_p)$ and $J_z(r_a, r_p)$.

Problem 2.4

a. Using a generic transformation $x^\alpha \rightarrow x'^\alpha$, establish the contravariant character of the coordinate differential dx^μ, and the covariant character of the gradient $\frac{\partial\phi(x)}{\partial x^\mu} \equiv \phi_{,\mu}$.

b. From the definition of polar coordinates in terms of Cartesian: $x = s\cos\phi$, $y = s\sin\phi$, construct the matrix-vector equation relating (dx, dy) to $(ds, d\phi)$. If we take the polar (s, ϕ) to be the transformed coordinates: $(x'^1 = s, x'^2 = \phi)$, and the Cartesian to be the original set $(x^1 = x, x^2 = y)$, show that your matrix-vector equation represents precisely the contravariant transformation rule.

c. Work out the gradient for the scalar $\psi = kxy$ in both Cartesian ($x^1 = x, x^2 = y$) and polar ($x'^1 = s, x'^2 = \phi$) coordinates. Show, by explicit construction, that the covariant transformation law relating $\psi'_{,\alpha}$ to $\psi_{,\alpha}$ holds in this case (i.e. start with $\psi_{,\mu}$ in Cartesian coordinates, transform according to $\psi'_{,\mu} = \frac{\partial x^\alpha}{\partial x'^\mu}\psi_{,\alpha}$ and show that this is what you get when you calculate $\psi'_{,\mu}$ explicitly).

Problem 2.5

Using the matrix point of view, construct the matrix form of both $\frac{\partial x^\alpha}{\partial x'^\beta}$ and $\frac{\partial x'^\beta}{\partial x^\gamma}$ for the explicit transformation from Cartesian to polar coordinates: ($x^1 = x, x^2 = y$), ($x'^1 = s, x'^2 = \phi$) (where $x = s\cos\phi$, $y = s\sin\phi$ defines (s, ϕ)). Write both transformation "matrices" in the original and new variables (so construct $\frac{\partial x^\alpha}{\partial x'^\beta}(x)$ and $\frac{\partial x^\alpha}{\partial x'^\beta}(x')$ and the similarly for $frac\partial x'^\alpha \partial x^\beta$). Verify, in both coordinate systems, that

$$\frac{\partial x^\alpha}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\gamma} = \delta^\alpha_\gamma. \tag{5}$$

With Leibniz notation, this relationship is clear, but work out the two matrices on the left and multiply them to see how this goes in real coordinate systems. The moral value of doing the matrix-matrix multiplication in both the original Cartesian, and the polar coordinate systems is to drive home the point that the relation is coordinate independent, but in order to obtain it correctly you must work in one or the other coordinates.

Problem 2.6

By combining tensors, we can form new objects with the correct tensorial character.

a. Take two first rank contravariant tensors f^μ and h^ν. If we form a direct product, $T^{\mu\nu} = f^\mu h^\nu$, we get a second rank contravariant tensor. By transforming f^μ and h^ν (for $x \longrightarrow x'$) in the product, write down the second rank contravariant tensor transformation law ($T'^{\mu\nu} =?$).

b. Do the same for the covariant second rank tensor constructed out of

f_μ and h_ν via $T_{\mu\nu} = f_\mu h_\nu$.

c. A scalar transforms as: $\phi'(x') = \phi(x(x'))$ (i.e. a transcription, no transformation). Show that by taking a contravariant f^α and covariant h_β, the product $\psi = f^\alpha h_\alpha$ is a scalar.

d. (Optional)

If $h_{\mu\nu}$ is a covariant second rank tensor, show that $h^{\mu\nu} \equiv (h_{\mu\nu})^{-1}$ (the matrix inverse) is a contravariant second rank tensor.