Problem Set 3

Classical Mechanics II
Physics 411

Due on September 21st 2007

Problem 3.1

We defined covariant differentiation in terms of contravariant vectors:

$$h^{\alpha}{}_{;\beta} = h^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\beta\sigma} h^{\sigma}. \tag{1}$$

If we demand that covariant differentiation, like "normal" partial derivatives, satisfy the product rule:

$$\left(p^{\alpha}\, q^{\beta}\right)_{;\gamma} = p^{\alpha}{}_{;\gamma}\, q^{\beta} + p^{\alpha}\, q^{\beta}{}_{;\gamma}, \tag{2}$$

then show that the covariant derivative of a covariant tensor must be:

$$h_{\alpha;\beta} = h_{\alpha,\beta} - \Gamma^{\sigma}_{\alpha\beta}\, h_{\sigma}. \tag{3}$$

Problem 3.2

a. It is possible to define geometries in which structure beyond the metric is needed. Consider the second (covariant) derivative of a scalar: $\phi_{;\mu\nu}$ – calculate the difference:

$$T_{\mu\nu} = \phi_{;\mu\nu} - \phi_{;\nu\mu}, \tag{4}$$

without using *any* known properties of the Christoffel connection. This difference is called the "torsion" of the geometry. What constraint must you place on the connection if cross-covariant-derivative equality is to hold (i.e. if the torsion vanishes)?

b. Show that if we require that $g_{\alpha\beta;\gamma} = 0$ and further that our geometry be torsion-free, then the Christoffel connection is related to the metric via:

$$\Gamma^{\rho}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}). \tag{5}$$

Problem 3.3

Finding the connection.

a. Calculate the Christoffel connection $\Gamma_{\alpha\beta\gamma}$ for three dimensional Euclidean space written in spherical coordinates.

b. Prove that a tensor that is zero in one coordinate system is zero in all coordinate systems.

c. Using the connection values in spherical coordinates as an example, prove that the connection is *not* a tensor.

Problem 3.4

Here we look at Killing vectors and associated constants.

a. Killing's equation is a covariant statement (meaning that it transforms appropriately under coordinate transformation):

$$f_{\mu;\nu} + f_{\nu;\mu} = 0. \tag{6}$$

If must, therefore, be true in *all* coordinate systems. We looked at the form of infinitesimal rotations in spherical coordinates, but the Cartesian form is even simpler – rotation through a small angle ω about the unit axis $\hat{\mathbf{\Omega}}$ can be expressed as:

$$\mathbf{x}' = \mathbf{x} + \omega \hat{\mathbf{\Omega}} \times \mathbf{x}. \tag{7}$$

Show that the associated infinitesimal transformation satisfies Killing's equation in Cartesian coordinates.

b. We know that the Hamiltonian for a transformation involving Killing vectors should be unchanged – consider the usual spherically symmetric Hamiltonian:

$$H = \frac{1}{2\,m} p_\alpha \, g^{\alpha\beta} \, p_\beta + U(r) \qquad r^2 \equiv x^\alpha \, g_{\alpha\beta} \, x^\beta, \tag{8}$$

in Cartesian coordinates. Find the momentum transformation $\mathbf{p}'$ associated with infinitesimal rotation, and construct the function:

$$H' = \frac{1}{2\,m} p'_\alpha g^{\alpha\beta} p'_\beta + U(r') \qquad r'^2 \equiv x'^\alpha g_{\alpha\beta} x'^\beta. \tag{9}$$

By inputting your expressions for $p'(x,p)$ and $x'(x,p)$, show that this is the same as H to first order in ω (note: you may assume that $g_{\mu\nu}$ is unchanged to first order here – i.e. don't worry about the transformation of the metric itself).

Problem 3.5

It is clear that $[H,H] = 0$, and so the Hamiltonian is itself a constant of any motion governed by the Hamiltonian. But what is the associated infinitesimal transformation? By treating H as an infinitesimal generator, find the transformation relating $(x'(t), p'(t))$ to $(x(t), p(t))$ and interpret it physically.