Problem Set 4

Classical Mechanics II
Physics 411

Due on September 28th 2007

Problem 4.1

Show, by explicit construction of the relevant Lorentz boost, that it is always possible to transform to the local rest frame of a massive particle moving along a worldline $x^\mu(t)$ (work in $D = 1 + 1$).

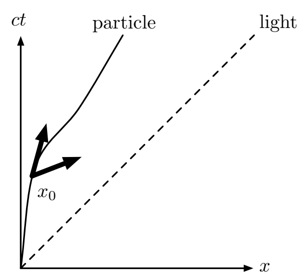

Figure 1: A particle is at x_0 – what is the Lorentz boost that transforms to the local rest frame of the particle at this point – write the transformation in terms of $v(x_0)$ (the velocity of the particle at x_0) and c.

Problem 4.2

Consider two connection fields, $C^\alpha_{\beta\gamma}$ and $\bar{C}^\alpha_{\beta\gamma}$ (not associated with a metric), find a linear combination of these two fields that is a tensor, or prove that no such combination exists.

Problem 4.3

a. The Kronecker delta is a tensor with numerical value:

$$\delta^\alpha_\beta = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}, \tag{1}$$

in Cartesian coordinates (say). Show that this numerical relation holds in *any* coordinate system.

b. For a second rank tensor $h_{\mu\nu}$ and the contravariant form defined by its matrix inverse, write the derivative $\frac{\partial h_{\mu\nu}}{\partial h_{\alpha\beta}}$ in terms of Kronecker deltas. How about $\frac{\partial h^{\mu\nu}}{\partial h_{\alpha\beta}}$? Note that this is how relations like (4.22) are developed.

Problem 4.4

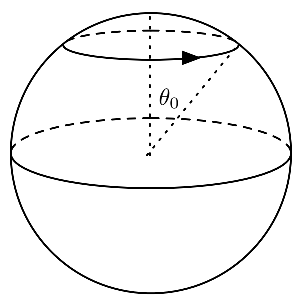

Figure 2: Sphere with path shown.

a. Consider the path, on the surface of a sphere, $(\theta = \theta_0, \phi = 0 \ldots 2\pi)$, i.e. we go around the sphere at angle θ_0 as shown in Figure 2. Construct a vector $f^\alpha(\phi)$ that is parallel-transported around this path starting at $\phi = 0$ with value (α and β are real here):

$$f^\alpha(\theta = \theta_0, \phi = 0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{2}$$

What is the magnitude of f^α along the curve (assume the sphere has radius $r = 1$)?

b. Take the two basis vectors at $(\theta = \theta_0, \phi = 0)$, i.e.

$$p^\alpha(\theta = \theta_0, \phi = 0) \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad q^\alpha(\theta = \theta_0, \phi = 0) \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{3}$$

show explicitly that at all points along the path from part a., the parallel-transported form of these initial vectors remain orthogonal. What is the angle that $p^\alpha(\phi = 0)$ makes with $p^\alpha(\phi = 2\pi)$?

Problem 4.5

A Killing vector was defined by considering the infinitesimal transformation that takes "coordinates to coordinates": $x'^\alpha = x^\alpha + \epsilon f^\alpha(x)$ with infinitesimal generator $J = p_\alpha f^\alpha$, and then requiring that $[H, J] = 0$ using $H = \frac{1}{2} p_\mu g^{\mu\nu} p_\nu$ as the free particle Hamiltonian. If we consider the "next step", we might allow linear mixing of momenta. Starting from:

$$x'^\alpha = x^\alpha + \epsilon p_\beta f^{\alpha\beta}(x), \tag{4}$$

a. Find the infinitesimal generator J that gives this transformation for x'^α. What is the expression (in terms of $f^{\alpha\beta}$) for the transformed momenta p'_α?

b. Construct the Poisson Bracket $[H, J]$ with your J and demand that it vanish to get the analogue of $f_{(\mu;\nu)} = 0$ (Killing's equation) for the tensor $f^{\mu\nu}(x)$. Be sure to write $[H, J] = 0$ in terms of manifestly tensorial objects, so that your final equation, like $f_{(\mu;\nu)} = 0$ is explicitly a tensor statement. The resulting equation is Killing's equation for second rank tensors, and such tensors are called "Killing tensors".

c. It should be obvious that $g^{\mu\nu}$ itself satisfies your tensorial Killing's equation. What is the conserved quantity J and the interpretation of the infinitesimal transformation if you set $f^{\mu\nu} = g^{\mu\nu}$?