Problem Set 5

Classical Mechanics II
Physics 411

Due on October 7th 2007

Problem 5.1

For the parabolic curve $y = x^2$ in two dimensions:

a. Find the curvature κ, plot this in the vicinity of zero.

b. Using the standard (p, e) parametrization of radius for an ellipse:

$$r(\phi) = \frac{p}{1 + e\,\cos\phi}, \tag{1}$$

find the curvature of the ellipse as a function of ϕ – plot for $\phi = 0 \rightarrow 2\,\pi$ with $p = 1$, $e = \frac{1}{2}$. Note - feel free to use Mathematica for this one, the derivatives can get . . . involved.

Problem 5.2

a. The Bianchi identity for the Riemann tensor was a cyclic relation involving the derivatives of $R_{\alpha\beta\gamma\delta}$. The electromagnetic field strength tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for four-potential A^μ. Find the analogue of the Bianchi identity for $F_{\mu\nu}$ (assume we're in flat Minkowski space) – i.e. a cyclic relation involving the derivatives of $F_{\mu\nu}$.

b. Using the Bianchi identity for the Riemann tensor, what is the relation between the gradient of the Ricci scalar, $R_{,\gamma}$, and the "divergence" of the Ricci tensor: $R^{\mu}{}_{\nu;\mu}$?

Problem 5.3

a. For a torus, parametrized by the two angles θ and ϕ as indicated in Figure 1 (with R the radius to the center of the tube, a the radius of the tube), find the metric, compute the connection coefficients, the Riemann $R^{\alpha}{}_{\beta\gamma\delta}$ and Ricci tensors and finally, the Ricci scalar.

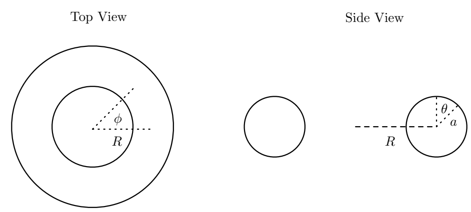

Figure 1: Toroidal surface parametrized by (R, a), ϕ and θ.

b. Our formula for the number of elements of the Riemann tensor indicates that in two dimensions, there should be only one independent element – why does it appear we ended up with two for the torus? What is the one independent element?

Problem 5.4

Find the Riemann tensor for the surface of an infinitely long cylinder (hint: do not calculate the Riemann tensor directly).

Problem 5.5

We have been using the implication:

$$g_{\mu\nu,\beta} = 0 \longrightarrow g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & \dots & \ddots \end{pmatrix}. \tag{2}$$

But, technically, all $g_{\mu\nu,\beta} = 0$ really implies is that $g_{\mu\nu}$ is coordinate independent (and we assume symmetric). Finish the argument: Prove that any symmetric, constant, real metric can be brought to the form above – diagonal with ones along the diagonal (assume purely spatial coordinates, so don't worry about possible temporal directions).