Problem Set 6

Classical Mechanics II
Physics 411

Due on October 12th 2007

Problem 6.1

Given a covariant second rank tensor $h_{\mu\nu}$ (with inverse $h^{\mu\nu}$, as usual), we can form its determinant by interpreting the tensor elements as matrix elements. One useful definition for $h \equiv \det(h_{\mu\nu})$ is:

$$h\,\delta^{\gamma}_{\alpha} = h_{\alpha\beta}\,C^{\beta\gamma}, \tag{1}$$

where $C^{\beta\gamma}$ is the cofactor matrix shown diagramatically below.

a. Using this, find the derivatives of the determinant w.r.t. the covariant components $h_{\mu\nu}$, i.e. what is $\frac{\partial h}{\partial h_{\mu\nu}}$ (hint: Check your result with a two-by-two matrix).

b. What about $\frac{\partial h}{\partial h^{\mu\nu}}$?

$$h_{\mu\nu} \doteq \begin{pmatrix} & h_{\alpha\beta} & \\ & & S^{\alpha\beta} \\ & & \end{pmatrix}$$

$$C^{\beta\gamma} \doteq \begin{pmatrix} |S^{11}| & -|S^{21}| & |S^{31}| & \cdots \\ -|S^{12}| & |S^{22}| & -|S^{32}| & \cdots \\ |S^{13}| & -|S^{23}| & |S^{33}| & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 1: The $S^{\alpha\beta}$ matrix, for an element $h_{\alpha\beta}$ is obtained by crossing out the row and column of $h_{\alpha\beta}$, collecting everything else into a $(D-1)\times(D-1)$ matrix. The matrix of cofactors is obtained by taking the determinant of $S^{\alpha\beta}$ and multiplying by $(-1)^{\beta-1}(-1)^{\gamma-1}$, arrayed as shown.

Problem 6.2

For a non-interacting dust at rest, we measure a uniform energy density ρ_0.

a. Suppose this energy density can be thought of as a number density (number of particles per unit volume) times the relativistic energy per particle. Compute the number density as observed in a frame moving along a shared x axis and multiply by the energy per particle as measured in that frame. This will give you the energy density as measured by the moving frame.

b. The quantity ρ_0 should transform as the 00 component of a second rank, symmetric tensor – in the rest frame of the dust, $T_{00} = \rho_0$ and all other components are zero. By Lorentz boosting along a shared x axis, find the component T'_{00} in a moving frame, it should be identical to your result above.

c. Show that the expression $\rho = T_{\mu\nu} u^\mu u^\nu$, where $T_{\mu\nu}$ is the stress tensor in the dust rest frame, and u^μ is the four-velocity of an observer (moving with speed v relative to the dust along a shared x axis) is correct for any observer (to do this, you need only verify that you get the result from the previous two parts for an observer moving with velocity v, and that you recover ρ_0 when you input the four-velocity of the stationary dust frame).

Problem 6.3

Given a Lagrange density for a scalar field ϕ in Minkowski space-time, find the field equations if $\mathcal{L}$ takes the form:

$$\mathcal{L}(\phi, \partial\phi, \partial\partial\phi), \tag{2}$$

that is, the Lagrangian depends on the field, its derivatives, and second derivatives (assume, here, that $\delta\phi$ vanishes on the boundary region, but also that $\delta\phi_{,\mu}$ vanishes).

Problem 6.4

a. Starting from the field equation for Newtonian gravity:

$$\nabla^2\phi = 4\pi G \rho_m \tag{3}$$

with ρ_m the mass density, rewrite in terms of ρ_e, energy density.

Suppose you now made the argument that $T_{\mu\nu}u^\mu u^\nu \sim \rho_e$, as we did in class, but without using $c = 1$ – what factor of c must you introduce so that $T_{\mu\nu}u^\mu u^\nu$ has units of energy density?

By following the usual identification, $R_{\mu\nu} \sim T_{\mu\nu}u^\mu u^\nu$, with all the units in place, write Einstein's equation *with the correct factors of* G *and* c in place. Note that the only thing you can't get out of this is the factor of 2 that takes the 4π above to 8π.

b. In units where $G = c = 1$, we can measure mass in meters – find the conversion factor α below:

$$M_{\text{in meters}} = \alpha\, M_{\text{in kilograms}}. \tag{4}$$

Using this factor, find the mass of the sun in meters.

Problem 6.5

The correct, scalar form of the action, yields field equations for a scalar ψ:

$$\frac{\partial}{\partial x^\mu}\left(g^{\mu\beta}\sqrt{-g}\psi_{,\beta}\right) = 0, \tag{5}$$

and it is clear that for Minkowski space-time represented in Cartesian coordinates, this yields the D'Alembertian:

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = 0. \tag{6}$$

Show that using cylindrical coordinates in Minkowski gives back the same equation (i.e. write out (5) explicitly using the Minkowski metric with cylindrical coordinates, and show that you can combine terms to form (6) where ∇^2 is the cylindrical Laplacian).