Problem Set 7

Classical Mechanics II
Physics 411

Due on October 26th 2007

Problem 7.1

A "massive" scalar field has action (Klein-Gordon):

$$S[\phi] = \int d\tau \sqrt{-g} \left[\frac{1}{2} \left(\phi_{,\alpha} g^{\alpha\beta} \phi_{,\beta} + m^2 \phi^2 \right) \right]. \tag{1}$$

Working in $D = 3 + 1$ Minkowski with Cartesian spatial coordinates:

a. Find the field equations and stress tensor $T^{\mu\nu}$ associated with this action.

b. Verify that $T^{\mu\nu}{}_{,\nu} = 0$ as a consequence of the field equations.

c. Write out the T^{00} component of the stress tensor, this is the energy density associated with the field.

d. Take a plane wave ansatz:

$$\phi(t, \mathbf{x}) = A e^{i p_\mu x^\mu} \tag{2}$$

for four-vector p_μ. What constraint does the field equation put on p_μ? Think of p_μ as the four-momentum of a particle, and explain why ϕ coming from (1) is called a "massive" scalar field.

Problem 7.2

The general scalar field Lagrangian (in Minkowski spacetime with Cartesian spatial coordinates) looks like:

$$\mathcal{L} = \frac{1}{2} \phi_{,\alpha} g^{\alpha\beta} \phi_{,\beta} - V(\phi). \tag{3}$$

a. Compute the general field equations for arbitrary $V(\phi)$. Write these explicitly for $D = 1+1$ Minkowski spacetime in Cartesian coordinates with[1] $V(\phi) = -\frac{m^2}{4v^2}(\phi^2 - v^2)^2$.

b. Our goal will be to generate the full solution from the stationary form. So set $\phi(x,t) = \phi(x)$ and solve the above field equations, assume $\phi'(x) = 0$ as $x \longrightarrow \pm\infty$ and $\phi = \pm v$ at infinity. Finally, you can take $\phi(x = 0) = 0$.

c. The full field equations are manifestly Lorentz invariant – what we'll do is put the Lorentz invariance back into our purely spatial solution by boosting out of a frame instantaneously at rest. Write x with $t = 0$ as a combination of x' and t' boosted along a shared x axis moving with velocity u. The resulting function $\phi(x(x',t')) \equiv \phi(x',t')$ *is* a solution to the full field equations.

Problem 7.3

Here we examine a re-working of scalar fields.

a. Write the scalar Lagrangian for two uncoupled massive scalar fields.

b. Show that this can be viewed as the scalar Lagrangian for a complex scalar field by varying $\mathcal{L} = \frac{1}{2}\phi_{,\mu}g^{\mu\nu}\phi^*_{,\nu} - V(\phi)$ (for appropriate $V(\phi)$) with respect to the real and complex parts separately.

Problem 7.4

For the time-independent scalar field equation $\nabla^2\phi(x,y,z) = 0$, write down a spherically symmetric ansatz for ϕ and solve the field equation.

Problem 7.5

Download the Mathematica package "EinsteinVariation.handout.m" and associated workbook. Using the definitions and functions you find there:

[1] Note that this potential is a natural non-linear extension of the Klein-Gordon field, for small v, we recover a mass term and a ϕ^4 term.

a. Verify that Minkowski space-time expressed in spherical coordinates is flat.

b. Find the curvature scalar (R) for the torus from your previous homework.

Problem 7.6

What is $F^{\mu}_{;\mu}$ (the four-divergence) of a vector field in Euclidean three dimensional space written in spherical coordinates. Did you get the expression in the front cover of Griffiths? If not, explain why and show how to recover the familiar expression.