Problem Set 8

Classical Mechanics II
Physics 411

Due on November 2nd, 2007

Problem 8.1

For the stress-tensor of E&M:

$$T^{\mu\nu} = \alpha \left[F^{\mu\sigma} F^{\nu}_{\ \sigma} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right], \tag{1}$$

a. Establish conservation using the source-free field equations (and any other identities you need).

b. What are the restrictions on a plane-wave ansatz:

$$A_\mu = K_\mu \, e^{i p_\mu x^\mu} \tag{2}$$

if this is to be a solution to the vacuum field equations in Lorentz gauge?

c. For the plane-wave solution, compute $T^{\mu\nu}$.

Problem 8.2

This problem begins to develop the GR analogue of the "Coulomb" solution from E&M (general spherically symmetric solution to Maxwell's equations). We saw, last week, how to impose spherical symmetry for a scalar field $\phi(x, y, z)$, now we extend to vectors and tensors.

a. Construct the most general spherically symmetric *vector* field $\mathbf{A}(\mathbf{r})$ (in what direction must it point?). Again, find the solution(s) to $\nabla^2 \mathbf{A} = 0$ in this setting.

b. Now we want to construct the most general spherically symmetric second rank tensor, $g_{ij}(\mathbf{r})$. This will end up being the spatial portion of the metric ansatz we will use to develop the so-called Schwarzschild solution. As a second rank symmetric tensor, your ansatz must have $g_{ij} = g_{ji}$, and as a portion of a spatial metric, *must* be invertible.

c. Take your expression for g_{ij} and form the line-element in "spherical coordinates" for your spherically symmetric space.

Problem 8.3

Write $F^{\mu\nu}F_{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ – this is a gauge-invariant form of the (scalar) Lagrangian for E&M. We can write this in terms of V and $\mathbf{A}$ using the identifications: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$. Assume spherical symmetry for V and $\mathbf{A}$ (with time included – spherical symmetry is a spatial statement, so you can still have arbitrary temporal dependence). Write your action (using $F^{\mu\nu}F_{\mu\nu}$ as your Lagrangian) with this spherical symmetry in place (i.e. use your spherically symmetric ansatz and write the action integrand in spherical coordinates), vary w.r.t. all unknown functions to find the field equations, and solve them. What you should get, in the end, is the potential V and $\mathbf{A}$ outside of a spherically symmetric source distribution.

Problem 8.4

a. Show that any two-dimensional space with positive definite metric (i.e. $g > 0$) has a metric that can (with appropriate coordinate choice) be written[1]:

$$g^{\mu\nu} \doteq f(X, Y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

with coordinates (X, Y). You may assume that the solution to the generalized Laplace's equation: $(\sqrt{g} g^{\mu\nu} \phi_{,\mu})_{,\nu} = 0$ exists – also assume that any metric is symmetric.

b. Construct an example of a two-dimensional space that has zero Ricci scalar, but non-zero Riemann tensor or show that no two-dimensional space has this property.

[1]This property is known as "conformal flatness", and can be established by a tensor test analagous to the Riemann tensor test for flatness. The relevant fourth rank tensor is known as the "Weyl tensor", and can be constructed from the trace-free part of the Riemann tensor, see, for example, D'Inverno pp. 87–89. In this problem, you should not use the Weyl tensor to establish conformal flatness.

Problem 8.5

Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = 8\,\pi\, T_{\mu\nu} \tag{4}$$

can be written in the equivalent form, in $D = 4$:

$$R_{\mu\nu} = 8\,\pi \left(T_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, T \right) \tag{5}$$

for $T \equiv T^{\alpha}_{\ \alpha}$, the trace of the stress-tensor. Show that this is true, and find the analogue of this equation for arbitrary dimension.