Problem Set 9

Classical Mechanics II
Physics 411

Due on November 9th, 2007

Problem 9.1

The Green's function for a (linear) differential operator $\mathcal{D}$ is defined to be the solution to $\mathcal{D}G(\mathbf{r}) = -\delta(\mathbf{r})$ (think of $\delta(\mathbf{r})$ as a point source at the origin). Find the (spherically symmetric) Green's function for the following operators (we are only interested in the solutions that die at infinity, except for ∇^2 in $D = 2$, there the interesting solution is infinite as $r \to \infty$) – all dimensions are spatial with Euclidean metric:

a. $D = 2$, $\mathcal{D} = \nabla^2$.

b. $D = 2$, $\mathcal{D} = \nabla^2 - \mu^2$ for $\mu \in \mathbb{R}$.

c. $D = 3$, $\mathcal{D} = \nabla^2$.

d. $D = 3$, $\mathcal{D} = \nabla^2 - \mu^2$ for $\mu \in \mathbb{R}$.

Problem 9.2

Continuing our work on the spherical metric ansatz.

a. Last week, you determined that, for a spatial tensor interpreted as a metric, spherical symmetry means $ds^2 = A(r)dr^2 + B(r)r^2(d\theta^2 + \sin^2\theta d\phi^2)$. Show that this can, with appropriate re-definition of r be reduced to $ds^2 = \tilde{A}(\tilde{r})d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)$. One can "finish" the job of specifying spherical symmetry in $D = 3 + 1$ by introducing a function $F(r)$ for g_{00}.

b. Show that in vacuum (i.e. for $T^{\mu\nu} = 0$), Einstein's equation reduces to $R_{\mu\nu} = 0$.

c. Using Mathematica (or by hand), compute and solve $R_{\mu\nu} = 0$ for your ansatz above. This is the Schwarzschild solution of Einstein's equations, appropriate to space-time outside of a spherically symmetric source.

d. Is your solution flat?

e. Show explicitly that $g_{\mu\nu;\alpha} = 0$ for your metric.

Problem 9.3

In three dimensions (two spatial, one temporal), there is an analogue of the trivial term (in four dimensions): $\epsilon^{\alpha\beta\gamma\delta} A_{\beta,\alpha} A_{\delta,\gamma}$ that is *not* a total divergence, and hence can change the field equations of E&M (such a form is called a "Chern-Simon" term).

a. Find this term, and by adding it to the usual E&M Lagrangian, find the field equations. You should work in Cartesian coordinates (in Minkowski space-time) to simplify life.

b. You have the source-free theory in the above, suppose we put a static (time-independent) point source at the origin of the spatial coordinates ($\rho = q\,\delta^2(\mathbf{r})$) – this gives a source for the right-hand side that looks like: $j^0 = q\,\delta^2(\mathbf{r})$ with $j^x = j^y = 0$. Solve this for the zero component of the vector A^μ (the "potential" in this theory – you may assume it is time-independent). Your solutions from the first problem will be useful.