Problem Set 10

Classical Mechanics II
Physics 411

Due on November 16th, 2007

Problem 10.1

When you computed the electromagnetic solution for the fields outside a spherically symmetric source, you worked directly from the action with spherical ansatz. We can use the same approach (known as the "Weyl Method") to find the Schwarzschild solution directly from the action.

a. The spherically symmetric metric ansatz has two undetermined functions, but we can parametrize it however we like. Starting from:

$$ds^2 = -a(r)\, b(r)^2\, dt^2 + \frac{1}{a(r)}\, dr^2 + r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (1)$$

compute the Ricci scalar.

b. Using the action $S_{EH} = \int d\tau\, \sqrt{-g} R$, form the integrand and vary w.r.t. $a(r)$ and $b(r)$ separately to obtain two field equations, solve them. You should recover the Schwarzschild metric.

c. What happens if you make the clever observation that $b(r) = 1$ in the action before you vary – start from the Ricci scalar with $b = 1$, and vary w.r.t. $a(r)$, does this work?

Problem 10.2

What is the metric for a three-dimensional sphere of radius r (add the angle ψ to θ and ϕ in order to define the coordinates)? Assuming that spherical symmetry for $D = 4 + 1$ means that we have separate radial functions scaling dt^2 and dr^2, and an unmodified S^3 portion, find the analogue of the Schwarzschild solution in $D = 4 + 1$ from Einstein's equation in vacuum.

Problem 10.3

We used arbitrary coordinate choice gauge invariance to make a trace-reversed metric perturbation with vanishing four-divergence. Show explicitly that the linearized Einstein tensor is insensitive to

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + f_{(\mu,\nu)}. \tag{2}$$

Problem 10.4

There is another natural "scalar" we could add to the Einstein action, one that is not R, and not quadratic (or higher) in R. Add this term and find the spherically symmetric solution to this modified theory of gravity (in $D = 3 + 1$).

Problem 10.5

For the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}, \tag{3}$$

find e^A using, at most, a calculator. You can check your answer using Mathematica's `MatrixExp` function.