Problem 3.1

For $h^{\alpha}{}_{;\beta} = h^{\alpha}{}_{,\beta} + \Gamma^{\alpha}_{\sigma\beta} h^{\sigma}$

we want $(p^{\alpha} q^{\beta})_{;\gamma} = p^{\alpha}{}_{;\gamma} q^{\beta} + p^{\alpha} q^{\beta}{}_{;\gamma}$

$p^{\alpha} q_{\alpha}$ is a scalar,

Then consider: $(p^{\alpha} q_{\alpha})_{;\gamma} = p^{\alpha}{}_{;\gamma} q_{\alpha} + p^{\alpha} q_{\alpha;\gamma} = p^{\alpha}{}_{,\gamma} q_{\alpha} + p^{\alpha} q_{\alpha,\gamma}$

using the definition for $p^{\alpha}{}_{;\gamma}$, we have

$$(p^{\alpha}{}_{,\gamma} + \Gamma^{\alpha}_{\gamma\sigma} p^{\sigma} - p^{\alpha}{}_{,\gamma}) q_{\alpha} = p^{\alpha}(q_{\alpha,\gamma} - q_{\alpha;\gamma})$$

or: $p^{\alpha}[q_{\alpha;\gamma} - q_{\alpha,\gamma} + \Gamma^{\beta}_{\gamma\alpha} q_{\beta}] = \emptyset.$

For arbitrary p^{α}, we have

$$\boxed{q_{\alpha;\gamma} = q_{\alpha,\gamma} - \Gamma^{\beta}_{\gamma\alpha} q_{\beta}}$$

Problem 3.2

a. $\Phi_{;\mu\nu} = (\Phi_{,\mu})_{;\nu}$ let $f_{\mu} \equiv \Phi_{,\mu}$, then:

$$\Phi_{;\mu\nu} = f_{\mu;\nu} = f_{\mu,\nu} - \Gamma^{\sigma}_{\mu\nu} f_{\sigma} = \Phi_{,\mu\nu} - \Gamma^{\sigma}_{\mu\nu} \Phi_{,\sigma}$$

so $$\Phi_{;\nu\mu} = \Phi_{,\nu\mu} - \Gamma^{\sigma}_{\nu\mu} \Phi_{,\sigma}$$

∴ the torsion is: $T_{\mu\nu} = \Phi_{;\mu\nu} - \Phi_{;\nu\mu} = -\Phi_{,\sigma}(\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu})$

For the torsion to vanish, we must have: $\Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$, i.e. a <u>symmetric</u> connection.

b.
$$\emptyset = g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \Gamma^{\sigma}_{\alpha\gamma} g_{\sigma\beta} - \Gamma^{\sigma}_{\beta\gamma} g_{\alpha\sigma}$$
$\beta \leftrightarrow \gamma$
$$\emptyset = g_{\alpha\gamma;\beta} = g_{\alpha\gamma,\beta} - \Gamma^{\sigma}_{\alpha\beta} g_{\sigma\gamma} - \Gamma^{\sigma}_{\gamma\beta} g_{\alpha\sigma}$$
$\alpha \leftrightarrow \beta.$
$$\emptyset = -g_{\beta\gamma;\alpha} = -g_{\beta\gamma,\alpha} + \Gamma^{\sigma}_{\beta\alpha} g_{\sigma\gamma} + \Gamma^{\sigma}_{\gamma\alpha} g_{\beta\sigma}$$

Add together w/ symmetric $\Gamma^{\mu}{}_{\nu\gamma} = \Gamma^{\mu}{}_{\gamma\nu}$ (& symmetric metric)

$$\emptyset = g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} - 2\Gamma^{\sigma}_{\beta\gamma} g_{\alpha\sigma}$$

so $\Gamma^{\sigma}_{\beta\gamma} g_{\alpha\sigma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})$ multiply both sides by $g^{\rho\alpha}$

$$\boxed{\Gamma^{\rho}_{\beta\gamma} = \frac{1}{2} g^{\rho\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})}$$

Problem 3.3

a. For spherical coordinates: $g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

&

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right)$$

start w/ $\alpha = 1$, then

$$\Gamma_{1\mu\nu} = \frac{1}{2}\left(g_{1\mu,\nu} + g_{1\nu,\mu} - g_{\mu\nu,1}\right)$$

then the non-zero entries are:

$$\Gamma_{122} = -\frac{1}{2}\frac{\partial g_{22}}{\partial r} = -r$$

and

$$\Gamma_{133} = -\frac{1}{2}\frac{\partial g_{33}}{\partial r} = -r\sin^2\theta$$

For $\alpha=2$: $\Gamma_{2\mu\nu} = \frac{1}{2}\left(g_{2\mu,\nu} + g_{2\nu,\mu} - g_{\mu\nu,2}\right)$

non-zero:

$$\Gamma_{212} = \Gamma_{221} = \frac{1}{2}\frac{\partial g_{22}}{\partial r} = r$$

$$\Gamma_{233} = -\frac{1}{2}\frac{\partial g_{33}}{\partial \theta} = -r^2\sin\theta\cos\theta$$

For $\alpha = 3$: $\Gamma_{3\mu\nu} = \frac{1}{2}\left(g_{3\mu,\nu} + g_{3\nu,\mu} - g_{\mu\nu,3}\right)$ no ϕ-dependence.

non-zero:

$$\Gamma_{313} = \Gamma_{331} = \frac{1}{2}\frac{\partial g_{33}}{\partial r} = r\sin^2\theta$$

$$\Gamma_{323} = \Gamma_{332} = \frac{1}{2}\frac{\partial g_{33}}{\partial \theta} = r^2\sin\theta\cos\theta.$$

b. For a tensor f^α, if we have $f^\alpha = \emptyset$ then in a new coordinate system $x \to x'$:

$$f'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} f^\beta = \emptyset \quad \left(\text{b/c } \frac{\partial x'^\alpha}{\partial x^\beta} \text{ is invertible, hence no null space to go other direction}\right)$$

so if f^α is zero in one coord. system, it is zero in all of them.

c. In spherical coordinates, $\Gamma_{\alpha\beta\gamma} \neq \emptyset$, while in Cartesian coords, $\Gamma_{\alpha\beta\gamma} = \emptyset$ so it is not a tensor.

Problem 3.4

a. In Cartesian coordinates,

$$f_{\mu;\nu} = f_{\mu,\nu} - \Gamma^{\sigma}_{\mu\nu} f_\sigma = f_{\mu,\nu}$$

$$\Gamma^{\sigma}_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) = \emptyset,$$

the metric has no spatial dependence.

Killing's equation reads: $f_{\mu;\nu} + f_{\nu;\mu} = \emptyset = f_{\mu,\nu} + f_{\nu,\mu}$.

For $\vec{x}' = \vec{x} + \omega \hat{\Omega} \times \vec{x}$, or

$$x'^{\alpha} = x^{\alpha} + \omega g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} x^{\nu}$$

we have

$$f^{\alpha} = g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} x^{\nu},$$

&

$$f_\alpha = \epsilon_{\alpha\mu\nu} \Omega^{\mu} x^{\nu}$$

then $f_{\alpha,\beta} = \frac{\partial f_\alpha}{\partial x^\beta} = \epsilon_{\alpha\mu\nu} \Omega^{\mu} \delta^{\nu}_{\beta} = \epsilon_{\alpha\mu\beta} \Omega^{\mu}$

&

$$\boxed{f_{\alpha,\beta} + f_{\beta,\alpha} = \epsilon_{\alpha\mu\beta} \Omega^{\mu} + \epsilon_{\beta\mu\alpha} \Omega^{\mu} = \epsilon_{\alpha\mu\beta} \Omega^{\mu} - \epsilon_{\alpha\mu\beta} \Omega^{\mu} = \emptyset}$$

(by $\epsilon_{\alpha\mu\beta} = -\epsilon_{\beta\mu\alpha}$), so Killing's equation is true, & f^{α} is a Killing vector.

b. For $f^{\alpha} = g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} x^{\nu}$, we have: $J = p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} x^{\nu}$

&

$$p'_\gamma = p_\gamma - \omega \frac{\partial J}{\partial x^\gamma} = p_\gamma - \omega p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} \delta^{\nu}_{\gamma}$$

$$= p_\gamma - \omega p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^{\mu}$$

then:

$$p'_\gamma g^{\gamma\nu} p'_\nu = (p_\gamma - \omega p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^{\mu})(p^{\gamma} - \omega p_\alpha g^{\alpha\beta} g^{\gamma\rho} \epsilon_{\beta\mu\rho} \Omega^{\mu})$$

$$= p_\gamma p^\gamma - \omega[p_\gamma p_\alpha g^{\alpha\beta} g^{\gamma\rho} \epsilon_{\beta\mu\rho} \Omega^{\mu} + p^{\gamma} p_\alpha g^{\alpha\beta} \epsilon_{\beta\mu\gamma} \Omega^{\mu}] + O(\omega^2)$$

$$= p_\gamma p^\gamma - \omega[\underbrace{p^\rho p^\beta}_{\text{sym. in } \rho\leftrightarrow\beta} \underbrace{\epsilon_{\beta\mu\rho}}_{\text{antisym. in } \rho\leftrightarrow\beta} \Omega^{\mu} + p^{\gamma} p^{\beta} \epsilon_{\beta\mu\gamma} \Omega^{\mu}] + O(\omega^2)$$

$$= p_\gamma g^{\alpha\gamma} p_\alpha + O(\omega^2).$$

each term in the bracket dies.

For r^2,

$$r'^2 = x'^{\alpha} g_{\alpha\beta} x'^{\beta} = (x^{\alpha} + \omega g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} x^{\nu})(x_\alpha + \omega \epsilon_{\alpha\mu\nu} \Omega^{\mu} x^{\nu})$$

$$= x^{\alpha} x_\alpha + \omega[\underbrace{x^{\alpha} x^{\nu} \epsilon_{\alpha\mu\nu} \Omega^{\mu}}_{=\emptyset} + \underbrace{g^{\alpha\beta} \epsilon_{\beta\mu\nu} \Omega^{\mu} x^{\nu} x_\alpha}_{=\emptyset}] + O(\omega^2)$$

$$= x^{\alpha} x_\alpha + O(\omega^2)$$

& $U(r'^2) = U(r^2) + O(\omega^2)$.

so $p'^{\alpha} p'_\alpha = p^{\alpha} p_\alpha + O(\omega^2)$ & $x'^{\alpha} x'_\alpha = x^{\alpha} x_\alpha + O(\omega^2)$

then

$$\boxed{H' = \tfrac{1}{2m} p'_\alpha p'^{\alpha} + U(r') = \tfrac{1}{2m} p_\alpha p^{\alpha} + U(r) + O(\omega^2) = H + O(\omega^2)}$$

Problem 3.5

We have $[H, H] = \emptyset$ trivially.

What are f^α & h_α in the infinitesimal transformation:

$$x'^\alpha = x^\alpha + \epsilon f^\alpha \qquad p'_\alpha = p_\alpha + \epsilon h_\alpha$$

when H is the generator?

$$f^\alpha(x,p) = \frac{\partial H}{\partial p_\alpha} \qquad h_\alpha(x,p) = -\frac{\partial H}{\partial x^\alpha}$$

We know, from the eqns of motion that $\frac{\partial H}{\partial p_\alpha} = \dot{x}^\alpha(t)$ & $\frac{\partial H}{\partial x^\alpha} = -\dot{p}_\alpha(t)$, so

$$x'^\alpha(t) = \underbrace{x^\alpha(t) + \epsilon \dot{x}^\alpha(t)}_{= x^\alpha(t+\epsilon) + O(\epsilon^2)} \quad \& \quad p'_\alpha(t) = \underbrace{p_\alpha(t) + \epsilon \dot{p}_\alpha(t)}_{p_\alpha(t+\epsilon) + O(\epsilon^2)}$$

so the new coordinates are, to $O(\epsilon^2)$ (& hence, beyond our interest):

$$x'^\alpha(t) = x^\alpha(t+\epsilon) \qquad p'_\alpha(t) = p_\alpha(t+\epsilon).$$

i.e. the transformation generated by H produces $x'(t)$ & $p'(t)$ that are the forward propagation in time of x & p.