Problem 1.2

For $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(\sqrt{x^2+y^2+z^2})$

for cylindrical coordinates, we have:
$$x = s\cos\phi$$
$$y = s\sin\phi$$
$$z = z$$

Then $\dot{x} = \dot{s}\cos\phi - s\sin\phi\,\dot{\phi}$ $\quad \dot{y} = \dot{s}\sin\phi + s\cos\phi\,\dot{\phi}$ $\quad \dot{z} = \dot{z}$

& the kinetic term becomes:

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{s}^2\cos^2\phi - 2s\dot{s}\cos\phi\sin\phi\,\dot{\phi} + \dot{s}^2\sin^2\phi + 2s\dot{s}\sin\phi\cos\phi\,\dot{\phi} + \dot{z}^2 + s^2\sin^2\phi\,\dot{\phi}^2 + s^2\cos^2\phi\,\dot{\phi}^2$$
$$= \dot{s}^2 + \dot{z}^2 + s^2\dot{\phi}^2$$

& for the potential term: $x^2 + y^2 + z^2 = s^2 + z^2$

Then

$$L = \frac{1}{2}m(\dot{s}^2 + \dot{z}^2 + s^2\dot{\phi}^2) - U(\sqrt{s^2+z^2})$$

the metric, setting $x^1 = s$, $x^2 = \phi$, $x^3 = z$ is

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 1.1

The two dimensional case suffices to show the pattern:

$$\mathbb{A}x = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A^{11}x_1 + A^{12}x_2 \\ A^{21}x_1 + A^{22}x_2 \end{pmatrix} = \begin{pmatrix} A^{1j}x_j \\ A^{2j}x_j \end{pmatrix}$$

so $\boxed{\mathbb{A}x \doteq A^{ij}x_j}$, & we recover the two elements by setting the open index $i = 1, 2$

$$x^T\mathbb{A} = (x_1 \;\; x_2)\begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} = \begin{pmatrix} x_1A^{11} + x_2A^{21} \\ x_1A^{12} + x_2A^{22} \end{pmatrix} = \begin{pmatrix} x_j A^{j1} \\ x_j A^{j2} \end{pmatrix} = \begin{pmatrix} A^{j1}x_j \\ A^{j2}x_j \end{pmatrix}$$

& $\boxed{x^T\mathbb{A} = A^{ji}x_j}$

Problem 1.3

a. $$T_{\mu\nu} = \underbrace{\tfrac{1}{2}[T_{\mu\nu}+T_{\nu\mu}]}_{\equiv S_{\mu\nu}} + \underbrace{\tfrac{1}{2}[T_{\mu\nu}-T_{\nu\mu}]}_{\equiv A_{\mu\nu}}$$

and we can verify: $S_{\nu\mu} = \frac{1}{2}[T_{\nu\mu}+T_{\mu\nu}] = S_{\mu\nu}$ ✓

$A_{\nu\mu} = \frac{1}{2}[T_{\nu\mu}-T_{\mu\nu}] = -A_{\mu\nu}$ ✓

b. $$T_{\mu\nu}Q^{\mu\nu} = S_{\mu\nu}Q^{\mu\nu} + \underbrace{A_{\mu\nu}Q^{\mu\nu}}_{\overset{!}{=} -A_{\nu\mu}Q^{\mu\nu} = -A_{\nu\mu}Q^{\nu\mu} = -A_{\mu\nu}Q^{\mu\nu}}$$

relabel dummy indices

& $A_{\mu\nu}Q^{\mu\nu} = -A_{\mu\nu}Q^{\mu\nu} \Rightarrow A_{\mu\nu}Q^{\mu\nu} = \emptyset.$

$$\boxed{T_{\mu\nu}Q^{\mu\nu} = S_{\mu\nu}Q^{\mu\nu}}$$

similarly, $\boxed{T_{\mu\nu}P^{\mu\nu} = \underbrace{S_{\mu\nu}P^{\mu\nu}}_{\emptyset} + A_{\mu\nu}P^{\mu\nu} = A_{\mu\nu}P^{\mu\nu}}$

c. Let $\dot{x}^\nu\dot{x}^\gamma \equiv S^{\nu\gamma}$, symmetric.

then (2) is: $S^{\nu\gamma}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2}\frac{\partial g_{\gamma\nu}}{\partial x^\alpha}\right] \equiv S^{\nu\gamma}T_{\alpha\nu\gamma}$

& only the portion of $T_{\alpha\nu\gamma}$ symmetric in $\nu \leftrightarrow \gamma$ will survive:

$$T_{\alpha\nu\gamma} = \underbrace{\tfrac{1}{2}[T_{\alpha\nu\gamma}+T_{\alpha\gamma\nu}]}_{} + \tfrac{1}{2}[T_{\alpha\nu\gamma}-T_{\alpha\gamma\nu}]$$

$$\overset{!}{=} \frac{1}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2}\frac{\partial g_{\gamma\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{1}{2}\frac{\partial g_{\nu\gamma}}{\partial x^\alpha}\right]$$

(since $g_{\gamma\nu} = g_{\nu\gamma}$)

$$= \frac{1}{2}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial g_{\gamma\nu}}{\partial x^\alpha}\right]$$

& $S^{\nu\gamma}T_{\alpha\nu\gamma} = \boxed{\frac{1}{2}\dot{x}^\nu\dot{x}^\gamma\left[\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\nu} - \frac{\partial g_{\gamma\nu}}{\partial x^\alpha}\right] = \dot{x}^\nu\dot{x}^\gamma\left[\frac{\partial g_{\alpha\nu}}{\partial x^\gamma} - \frac{1}{2}\frac{\partial g_{\gamma\nu}}{\partial x^\alpha}\right]}$

Problem 1.4

a. $$m g_{\alpha\nu}\ddot{x}^{\nu} + \tfrac{1}{2} m \dot{x}^{\nu}\dot{x}^{\gamma}\left[\frac{\partial g_{\alpha\nu}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\nu}} - \frac{\partial g_{\gamma\nu}}{\partial x^{\alpha}}\right] = -\frac{\partial U}{\partial x^{\alpha}}$$

For $\alpha=1$, corresponding to the r coordinate:

$$m g_{1\nu}\ddot{x}^{\nu} + \tfrac{1}{2} m \dot{x}^{\nu}\dot{x}^{\gamma}\left[\frac{\partial g_{1\nu}}{\partial x^{\gamma}} + \frac{\partial g_{1\gamma}}{\partial x^{\nu}} - \frac{\partial g_{\gamma\nu}}{\partial x^{1}}\right] = -\frac{\partial U}{\partial x^{1}}. \qquad (*)$$

the metric, in spherical coordinates, is:

$$g_{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

so the only non-zero terms are g_{11}, g_{22} & g_{33}. Then (*) is

$$m\ddot{x}^1 + \tfrac{1}{2} m\left(-\dot{x}^2\dot{x}^2\frac{\partial g_{22}}{\partial x^1} - \dot{x}^3\dot{x}^3\frac{\partial g_{33}}{\partial x^1}\right) = -\frac{\partial U}{\partial x^1}.$$

or, using $x^1=r$, $x^2=\theta$, $x^3=\phi$:

$$m\ddot{r} + \tfrac{1}{2} m\left[-\dot{\theta}^2 \cdot 2r - \dot{\phi}^2 \cdot 2r\sin^2\theta\right] = -\frac{\partial U}{\partial r}$$

$$\boxed{m\ddot{r} - m\left[r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right] = -\frac{\partial U}{\partial r}}$$

b. Starting from $L = \tfrac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - U(r)$
we have:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \varnothing$$

$$\frac{d}{dt}\left[m\dot{r}\right] - \tfrac{1}{2} m\left(2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2\right) + \frac{\partial U}{\partial r} = \varnothing.$$

or

$$m\ddot{r} - m\left[r\dot{\theta}^2 + r\sin^2\theta\,\dot{\phi}^2\right] = -\frac{\partial U}{\partial r}$$ as above.