Problem 6.1

a. Given $\delta^\gamma_\alpha h = h_{\alpha\beta} C^{\beta\gamma}$ for $h \equiv \det(h_{\mu\nu})$.
we know that $C^{\beta\gamma}$ is independent of $h_{\alpha\beta}$, so

$$\frac{\partial}{\partial h_{\alpha\beta}}(\delta^\gamma_\alpha h) = C^{\beta\gamma} \longrightarrow \delta^\gamma_\alpha \frac{\partial h}{\partial h_{\alpha\beta}} = C^{\beta\gamma}$$

multiply both sides by $h_{\sigma\beta}$, & we have

$$h_{\sigma\beta} \frac{\partial h}{\partial h_{\gamma\beta}} = \underbrace{h_{\sigma\beta} C^{\beta\gamma}}$$

$$h_{\sigma\beta} \frac{\partial h}{\partial h_{\gamma\beta}} = h \delta^\gamma_\sigma$$

now using $h^{\rho\sigma} h_{\sigma\beta} = \delta^\rho_\beta$, we have:

$$\underbrace{\delta^\rho_\beta \frac{\partial h}{\partial h_{\gamma\beta}}} = h^{\rho\sigma} h \delta^\gamma_\sigma$$

$$\boxed{\frac{\partial h}{\partial h_{\gamma\rho}} = h^{\rho\gamma} h}$$

b. For $h^{\mu\nu}$, w/ $\tilde{h} \equiv \det(h^{\mu\nu})$, we get by identical argument:

$$\frac{\partial \tilde{h}}{\partial h^{\gamma\rho}} = h_{\rho\gamma} \tilde{h}$$

& we know that $\tilde{h} = 1/h$, so

$$\frac{\partial (1/h)}{\partial h^{\gamma\rho}} = \frac{1}{h} h_{\rho\gamma}$$

$$-1/h^2 \frac{\partial h}{\partial h^{\gamma\rho}} = \frac{1}{h} h_{\rho\gamma}$$

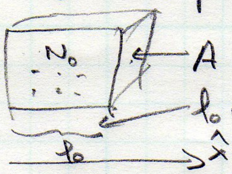
so

$$\boxed{\frac{\partial h}{\partial h^{\gamma\rho}} = -h h_{\rho\gamma}}$$

Problem 6.3

a. For $\rho_0 = N_0 m$ w/ N_0 the number density (number of particles per unit vol in the rest frame of the dust, & m representing mc^2, the rest energy of each particle.

we have:



The box shrinks by γ^{-1} for an observer moving along the $\hat{x}$ axis w/ w/ speed v:

$$l = \frac{1}{\gamma} l_0 = l_0 \sqrt{1-v^2} \quad \& \text{ then } N = \frac{N_0}{\sqrt{1-v^2}} .$$

To the observer, the particles have energy $E = \frac{m}{\sqrt{1-v^2}}$ (they have rest energy m, but also a kinetic portion since they appear to be moving w/ speed v). So the energy density, measured by the observer is:

$$\rho = N \cdot E = \frac{N_0 m}{(1-v^2)} = \frac{\rho_0}{(1-v^2)} .$$

b. For a rest stress-energy tensor of the form:

$$T_{\mu\nu} = \delta^0_\mu \delta^0_\nu \rho_0$$

& a Lorentz boost in the x-direction: $\Lambda^\mu{}_\alpha \doteq \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

we have

$$T'_{\alpha\beta} = T_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \quad \text{so} \quad T'_{00} = \rho_0 \delta^0_\mu \delta^0_\nu \Lambda^\mu{}_0 \Lambda^\nu{}_0$$
$$= \rho_0 (\Lambda^0{}_0)^2 .$$

or $$\rho' = \frac{\rho_0}{(1-v^2)} . \quad \text{as above.}$$

c. In the rest frame, we have $V^\mu \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \delta^\mu_0$, so $T_{\mu\nu} V^\mu V^\nu = T_{00} = \rho_0$

For the observer, $V'^\mu = \Lambda^\mu{}_\alpha V^\alpha$ & $T_{\mu\nu} V'^\mu V'^\nu = T_{\mu\nu} \Lambda^\mu{}_\alpha V^\alpha \Lambda^\nu{}_\beta V^\beta$
$$= T_{\mu\nu} \Lambda^\mu{}_0 \Lambda^\nu{}_0$$
$$= \rho_0 (\Lambda^0{}_0)^2$$

so $\boxed{T_{\mu\nu} V'^\mu V'^\nu = \frac{\rho_0}{(1-v^2)}}$ as desired.

Problem 6.3

$$S[\phi+\delta\phi] = \int \mathcal{L}(\phi+\delta\phi, \phi_{,\mu}+\delta\phi_{,\mu}, \phi_{,\mu\nu}+\delta\phi_{,\mu\nu})\,d\tau$$

$$= \int \Big[\mathcal{L}(\phi, \phi_{,\mu}, \phi_{,\mu\nu}) + \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi_{,\mu} + \frac{\partial\mathcal{L}}{\partial\phi_{,\mu\nu}}\delta\phi_{,\mu\nu}}_{\delta S} \Big] d\tau.$$

Now:
$$\int_\Omega \left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi\right)_{,\mu} d\tau = \int_{\partial\Omega}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi\right) da^\mu = \emptyset$$

$\delta\phi$ vanishes on the border

$$= \int \left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi_{,\mu} + \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right)\right]\delta\phi \right) d\tau.$$

So we have the usual integration-by-parts:

$$\int \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\delta\phi_{,\mu}\,d\tau = -\int \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right)\right]\delta\phi\,d\tau$$

And for the second derivatives, we integrate by parts twice:

$$\int \frac{\partial\mathcal{L}}{\partial\phi_{,\mu\nu}}\delta\phi_{,\mu\nu}\,d\tau = -\left[-\int\left[\partial_\mu\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial\phi_{,\mu\nu}}\right]\right]\delta\phi\,d\tau\right]$$

(assuming $\delta\phi_{,\mu}$ vanishes on the boundary).

Putting all the terms together, we have:

$$\delta S = \int\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu\nu}}\right)\right]\delta\phi\,d\tau$$

For $\delta S=\emptyset$ to be true for arbitrary $\delta\phi$, we must have:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu\nu}}\right) = \emptyset}$$

Problem 6.4

a. If we start w/ Newtonian field:

$$\nabla^2 \Phi = 4\pi G \rho_m$$

w/ ρ_m the mass density, then we can rewrite in terms of energy density:

$$E = mc^2 \longrightarrow \rho_E = \rho_m c^2$$

so

$$\nabla^2 \Phi = 4\pi \frac{G}{c^2} \rho_E .$$

Now $T_{00} = \rho_E$, energy density, so in order to make the energy density $T_{\mu\nu} u^\mu u^\nu$ observed by a lab moving w/ four-velocity u^μ have units of energy density, we must multiply by $\frac{1}{c^2}$:

$$\rho_E^{obs.} = \frac{1}{c^2} T_{\mu\nu} u^\mu u^\nu .$$

Now when we associate: $u^\mu u^\nu R_{\mu\nu} \sim \nabla^2 \Phi$
we will have:

$$u^\mu u^\nu R_{\mu\nu} \sim \underbrace{4\pi\, G/c^2\, \rho_E}$$

$$= 4\pi\, G/c^4\, T_{\mu\nu} .$$

and then we pick up the usual factor of 2:

$$\left(R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R\right) = 8\pi\, G/c^4\, T_{\mu\nu}$$

b. We want the conversion factor that take M in kg to M in meters: This must be made out of $1 = G^p c^q$ since G & c are 1.:

so: $m = kg \cdot \left(\frac{Nm^2}{kg^2}\right)^p \cdot \left(m/s\right)^q = kg \left(\frac{m^3}{kg\, s^2}\right)^p \left(m/s\right)^q = kg^{1-p} \cdot m^{3p+q}\, s^{-2p-q}$

& we have: $p = 1$, $3p + q = 1 \Rightarrow q = -2$

so $M_m = G/c^2\, M_{kg}$

& for the sun, $M_{kg} = 1.99 \times 10^{30}$, so:

$$M_m = \frac{6.67 \times 10^{-11} \frac{m^3}{kg\, s^2}}{(3 \times 10^8\, m/s)^2} \cdot 1.99 \times 10^{30}\, kg \simeq 1475\, m \approx 1.48\, km$$

Problem 6.5

For: $\frac{\partial}{\partial x^\mu}\left(g^{\mu\beta}\sqrt{-g}\,\psi_{,\beta}\right) = \emptyset$

in cylindrical coordinates, we have: $g_{\mu\nu} \doteq \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

w/ $dx^\mu \doteq \begin{pmatrix} c\,dt \\ ds \\ d\phi \\ dz \end{pmatrix}$

Then $\sqrt{-g} = s$

$$\frac{\partial}{\partial x^\mu}\left(g^{\mu\beta}\sqrt{-g}\,\frac{\partial\psi}{\partial x^\beta}\right) = \frac{1}{c}\frac{\partial}{\partial t}\left[-s\frac{1}{c}\frac{\partial\psi}{\partial t}\right] + \frac{\partial}{\partial s}\left[s\frac{\partial\psi}{\partial s}\right] + \frac{\partial}{\partial\phi}\left[\frac{1}{s^2}\cdot s\frac{\partial\psi}{\partial\phi}\right] + \frac{\partial}{\partial z}\left[s\frac{\partial\psi}{\partial z}\right]$$

$$= -\frac{1}{c^2}s\frac{\partial^2\psi}{\partial t^2} + \frac{\partial}{\partial s}\left(s\frac{\partial\psi}{\partial s}\right) + \frac{1}{s}\frac{\partial^2\psi}{\partial\phi^2} + s\frac{\partial^2\psi}{\partial z^2} = \emptyset$$

$$= -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \underbrace{\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\psi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}}_{=\nabla^2\psi \text{ in cylindrical coordinates}} = \emptyset.$$