Problem 7.1

a. A massive scalar field has:

$$S = \int d^4x \left[\left(\tfrac{1}{2}\Phi_{,\alpha} g^{\alpha\beta}\Phi_{,\beta}\right) + \tfrac{1}{2}m^2\Phi^2\right]\sqrt{-g}$$

w/ field equations: (in Cartesian coordinates).

$$\left(\frac{\partial\mathcal{L}}{\partial\Phi_{,\mu}}\right)_{,\mu} - \frac{\partial\mathcal{L}}{\partial\Phi} = \emptyset$$

$$\boxed{\Phi_{,}{}^{\mu}{}_{\mu} - m^2\Phi = \emptyset} \qquad (*).$$

The stress tensor is given by:

$$T^{\mu\nu} = -\left[g^{\mu\nu}\mathcal{L} + 2\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\right] \qquad \left(\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -g^{\alpha\mu}g^{\nu\beta}\right)$$

$$= -\left[g^{\mu\nu}\left(\tfrac{1}{2}\Phi_{,\alpha}g^{\alpha\beta}\Phi_{,\beta} + \tfrac{1}{2}m^2\Phi^2\right) - \Phi_{,\alpha}\Phi_{,\beta}g^{\alpha\mu}g^{\beta\nu}\right]$$

b. $$T^{\mu\nu}{}_{,\nu} = -\left[g^{\mu\nu}\left(\Phi_{,\alpha\nu}g^{\alpha\beta}\Phi_{,\beta} + m^2\Phi\Phi_{,\nu}\right) - \left(\Phi_{,\alpha\nu}\Phi_{,\beta} + \Phi_{,\alpha}\Phi_{,\beta\nu}\right)g^{\alpha\mu}g^{\beta\nu}\right]$$

$$= -\left[\underbrace{\Phi_{,}{}^{\mu}{}_{\alpha}\Phi_{,}{}^{\alpha} + m^2\Phi\Phi_{,}{}^{\mu} - \Phi_{,}{}^{\mu\nu}\Phi_{,\nu}}_{\text{cancel.}} - \Phi_{,}{}^{\mu}\Phi_{,}{}^{\nu}{}_{\nu}\right]$$

$$= -\left[m^2\Phi\Phi_{,}{}^{\mu} - \Phi_{,}{}^{\nu}{}_{\nu}\Phi_{,}{}^{\mu}\right]$$

or $$T^{\mu\nu}{}_{,\nu} = \Phi_{,}{}^{\mu}\left[\Phi_{,}{}^{\nu}{}_{\nu} - m^2\Phi\right] = \emptyset$$

by field eqn.s (*)

c. $$T^{00} = -\left[-\tfrac{1}{2}\left(-\dot{\phi}^2 + \nabla\phi\cdot\nabla\phi + m^2\phi^2\right) - \dot{\phi}^2\right]$$

$$\boxed{T^{00} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\nabla\phi\cdot\nabla\phi + \tfrac{1}{2}m^2\phi^2}$$

d. For a plane wave ansatz, $\phi(t,\vec{x}) = Ae^{ip_\mu x^\mu}$, the field equation gives:

$$\Phi_{,\mu}{}^{\mu} = -p_\mu p^\mu\Phi \quad \& \quad \Phi_{,\mu}{}^{\mu} - m^2\Phi = \emptyset \Rightarrow \boxed{-p_\mu p^\mu - m^2 = \emptyset}$$

or, in units w/ c=1: $-\left(-p_0^2 + \vec{p}\cdot\vec{p}\right) = m^2$

$$\boxed{E^2 - p^2 = m^2} \leftarrow \text{energy-momentum relation for}$$

a massive particle.

Problem 7.2

a. For $\mathcal{L} = \frac{1}{2} \Phi_{,\alpha} g^{\alpha\beta} \Phi_{,\beta} - V(\Phi)$

the field equations: $\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = \emptyset$
read:

$$\Phi_{,}{}^{\mu}{}_{\mu} + \frac{\partial V}{\partial \Phi} = \emptyset.$$

In $D=1+1$ Minkowski spacetime, & $V(\Phi) = \frac{-m^2}{4V^2}(\Phi^2 - V^2)^2$, this becomes:

$$\boxed{\left[-\ddot{\Phi} + \Phi''\right] - \frac{m^2}{V^2}(\Phi^2 - V^2)\Phi = \emptyset}$$

for $\Phi(x,t)$ w/ $\dot{\Phi} \equiv \frac{\partial \Phi}{\partial t}$, $\Phi' = \frac{\partial \Phi}{\partial x}$.

b. Consider a stationary limit:

$$\Phi'' - \frac{m^2}{V^2}(\Phi^2 - V^2)\Phi = \emptyset.$$

This has: $\frac{1}{2}\frac{d}{dx}\Phi'^2 = +\frac{m^2}{V^2}\frac{1}{4}\frac{d}{dx}(\Phi^4) - m^2\frac{1}{2}\frac{d}{dx}(\Phi^2)$

Then integrating once gives:

$$\frac{1}{2}\Phi'^2 = +\frac{m^2}{4V^2}\Phi^4 - \frac{m^2}{2}\Phi^2 + \alpha$$

At spatial infinity, we have $\Phi' = \emptyset$ & $\Phi = \pm V$, so there, this equation becomes:

$$\alpha = -\frac{m^2}{4V^2}V^4 + \frac{m^2}{2}V^2 = +\frac{1}{4}V^2 m^2$$

& we can write:

$$\frac{1}{2}\Phi'^2 = +\frac{m^2}{4V^2}(\Phi^2 - V^2)^2$$

$$\Phi' = \pm\frac{m}{\sqrt{2}V}(\Phi^2 - V^2)$$

take the + solution. Now we can solve the above - take

$$\Phi(x) = A\tanh(Bx), \quad \Phi' = \frac{AB}{\sinh^2(Bx)}$$

& we want:

$$\frac{AB}{\cosh^2(Bx)} = \frac{m}{\sqrt{2}V}\left(\frac{A^2\sinh^2(Bx) - V^2\cosh^2(Bx)}{\cosh^2(Bx)}\right)$$

$A = V$, & $\sinh^2(Bx) - \cosh^2(Bx) = -1$

$$\frac{VB}{\cosh^2(Bx)} = -\frac{mV}{\sqrt{2}V}\frac{1}{\cosh^2(Bx)} \Rightarrow B = -\frac{m}{\sqrt{2}}$$

Problem 7.2 (continued)

So $\boxed{\Phi(x) = v \tanh\left(-\frac{m}{\sqrt{2}} x\right)}$

c. We imagine boosting to a moving frame w/ speed u - then

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} +\gamma & -\gamma u \\ -\gamma u & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix}$$

& we have $t' = -\gamma u x \quad x' = \gamma x$, or, combining,

$$x = -\gamma u t' + \gamma x', \text{ w/ } \gamma = \frac{1}{\sqrt{1-u^2}}.$$

In the new frame, we require $\Phi(x,t) = \Phi(x',t')$

$$\Phi(x) = V\tanh\left(-\frac{m}{\sqrt{2}}\left(\frac{x'-ut'}{\sqrt{1-u^2}}\right)\right)$$

So we expect: $\boxed{\Phi(x,t) = v\tanh\left(-\frac{m}{\sqrt{2}}\left(\frac{x-ut}{\sqrt{1-u^2}}\right)\right)}.$

Problem 7.3

a. A single massive scalar field has Lagrange density:

$$\mathcal{L}_1 = \int \left(\tfrac{1}{2}\Phi_{,\alpha} g^{\alpha\beta}\Phi_{,\beta} + \tfrac{1}{2}m^2\Phi^2\right) d\tau \quad \text{in Cartesian coordinates}$$

leading to field equation

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}_1}{\partial \Phi_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0$$

$$\Phi_{,}{}^{\mu}{}_{\mu} - m^2\Phi = 0$$

or, in $D = 1+1$

$$-\dot{\phi}^2 + \phi'^2 - m^2\Phi = 0$$

To generate 2 scalar fields that do not interact:

$$\mathcal{L}_2 = \int \tfrac{1}{2}\left[u_{,\alpha} g^{\alpha\beta} u_{,\beta} + v_{,\alpha} g^{\alpha\beta} v_{,\beta} + m^2u^2 + m^2v^2\right] d\tau$$

& this will give $\quad u_{,}{}^{\mu}{}_{\mu} = m^2 u \quad$ & $\; v_{,}{}^{\mu}{}_{\mu} = m^2 v$

when varied w.r.t. u & v separately.

b. To repackage into a complex field $\Phi \equiv a + ib$:

$$\mathcal{L} = \tfrac{1}{2}\Phi_{,\alpha} g^{\alpha\beta}\Phi^*_{,\beta} + \tfrac{1}{2}m^2\Phi\Phi^*$$

$$\overset{!}{=} \tfrac{1}{2}(a_{,\alpha} + ib_{,\alpha}) g^{\alpha\beta}(a_{,\beta} - ib_{,\beta}) + \tfrac{1}{2}m^2(a^2 + b^2)$$

$$\overset{!}{=} \tfrac{1}{2}a_{,\alpha} g^{\alpha\beta} b_{,\beta} + \tfrac{1}{2}b_{,\alpha} g^{\alpha\beta} b_{,\beta} + \tfrac{1}{2}m^2(a^2 + b^2)$$

we just identify: $\Phi = u + iv$.

Problem 7.4

For a generic scalar $\Phi(x,y,z)$, spherical symmetry means that Φ is constant on spheres, so depends on $x, y, + z$ only through the particular combination

$$r = \sqrt{x^2+y^2+z^2}.$$

so $\Phi(x,y,z) = \Phi(\sqrt{x^2+y^2+z^2}) = \Phi(r)$.

The Laplacian, in spherical coordinates, applied to a function w/ no angular dependence is:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{d\Phi}{dr}\right)$$

$$\nabla^2\Phi = 0 \Rightarrow r^2\frac{d\Phi}{dr} = \alpha \quad \text{for constant } \alpha.$$

Then

$$\Phi' = \frac{\alpha}{r^2} \Rightarrow \boxed{\Phi(r) = -\frac{\alpha}{r} + \beta}$$

Problem 7.6

$$F^\mu{}_{;\mu} = F^\mu{}_{,\mu} + \Gamma^\mu{}_{\sigma\mu} F^\sigma$$

we know the connection for spherical coordinates (see Problem 3.3)

$$\Gamma_{r\theta\theta} = -r \quad \Gamma_{r\phi\phi} = -r\sin^2\theta$$

$$\Gamma_{\theta r\theta} = r \quad \Gamma_{\theta\phi\phi} = -r^2\sin\theta\cos\theta$$

$$\Gamma_{\phi r\phi} = r\sin^2\theta \quad \Gamma_{\phi\theta\phi} = r^2\sin\theta\cos\theta$$

so $\Gamma^\mu{}_{\sigma\mu} = g^{\alpha\mu}\Gamma_{\alpha\sigma\mu} = g^{rr}\Gamma_{r\sigma r} + g^{\theta\theta}\Gamma_{\theta\sigma\theta} + g^{\phi\phi}\Gamma_{\phi\sigma\phi}$

$$\Gamma^\mu{}_{\sigma\mu}F^\sigma = g^{\theta\theta}\Gamma_{\theta r\theta}F^r + g^{\phi\phi}\Gamma_{\phi r\phi}F^r + g^{\phi\phi}\Gamma_{\phi\theta\phi}F^\theta$$

$$= \frac{1}{r^2}(rF^r) + \frac{1}{r^2\sin^2\theta}(r\sin^2\theta)F^r + \frac{1}{r^2\sin^2\theta}\cdot r^2\sin\theta\cos\theta F^\theta$$

$$\boxed{F^\mu{}_{;\mu} = \frac{\partial F^r}{\partial r} + \frac{\partial F^\theta}{\partial\theta} + \frac{\partial F^\phi}{\partial\phi} + \frac{1}{r}F^r + \frac{1}{r}F^r + \frac{\cos\theta}{\sin\theta}F^\theta} \leftarrow$$

our basis vectors are $\vec{e}_r = \hat{r}$, $\vec{e}_\theta = r\hat{\theta}$ + $\vec{e}_\phi = r\sin\theta\,\hat{\phi}$

so $\vec{F} = F^\mu\vec{e}_\mu = F^r\vec{e}_r + F^\theta\vec{e}_\theta + F^\phi\vec{e}_\phi = F^r\hat{r} + F^\theta r\hat{\theta} + F^\phi r\sin\theta\hat{\phi}$

then

$$\nabla\cdot\begin{pmatrix}F^r\\ rF^\theta\\ r\sin\theta F^\phi\end{pmatrix} = \frac{1}{r^2}\left(2rF^r + r^2\frac{\partial F^r}{\partial r}\right) + \frac{1}{r\sin\theta}\left(\cos\theta F^\theta r + \sin\theta\, r\frac{\partial F^\theta}{\partial\theta}\right)$$

$$+ \frac{1}{r\sin\theta}\cdot r\sin\theta\frac{\partial F^\phi}{\partial\phi}$$

$$= \boxed{\frac{\partial F^r}{\partial r} + \frac{\partial F^\theta}{\partial\theta} + \frac{\partial F^\phi}{\partial\phi} + \frac{2}{r}F^r + \frac{\cos\theta}{\sin\theta}F^\theta} \leftarrow$$

In[1]:=
```
<< /Users/jfrankli/bin/EinsteinVariation.handout.m
```

Problem 7.5

Part a.

In[5]:=
```
gll = {{-1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, r^2 , 0}, {0, 0, 0, r^2 Sin[T]^2}};
Xu = {t, r, T, P};
Simplify[GetRiemann[gll, Xu]]
```

Out[7]=
```
{{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
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  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}},
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 {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
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  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}}
```

Part b.

In[8]:=
```
gll = {{a^2, 0}, {0, (R + a Sin[T])^2}};
Xu = {T, P};
GetRicciS[gll, Xu]
```

Out[10]=
$$\frac{2\,\mathrm{Sin}[T]}{a\,(R + a\,\mathrm{Sin}[T])}$$