Problem 8.1

a. $T^{\mu\nu}{}_{,\nu} = \alpha[F^{\mu\sigma}{}_{,\nu}F^{\nu}{}_{\sigma} + F^{\mu\sigma}F^{\nu}{}_{\sigma,\nu} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}{}_{,\nu}F_{\alpha\beta} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta,\nu}]$

The source-free field equations read:

$$F^{\mu\nu}{}_{,\nu} = \emptyset$$

So the second term is automatically zero.

$$= \alpha[F^{\mu\sigma}{}_{,}{}^{\nu}F_{\nu\sigma} - \frac{1}{4}F^{\alpha\beta}{}_{,}{}^{\mu}F_{\alpha\beta} - \frac{1}{4}F^{\alpha\beta}{}_{,}{}^{\mu}F_{\alpha\beta}]$$

$$= \alpha[F^{\mu\sigma}{}_{,}{}^{\nu} - \frac{1}{2}F^{\nu\sigma}{}_{,}{}^{\mu}]F_{\nu\sigma}$$

antisymmetrization in $\nu \leftrightarrow \sigma$

$$= \frac{1}{2}\alpha[F^{\mu\sigma}{}_{,}{}^{\nu} - F^{\mu\nu}{}_{,}{}^{\sigma} - F^{\nu\sigma}{}_{,}{}^{\mu}]F_{\nu\sigma}$$

The Bianchi identity reads: $F^{\mu\sigma}{}_{,}{}^{\nu} + F^{\nu\mu}{}_{,}{}^{\sigma} + F^{\sigma\nu}{}_{,}{}^{\mu} = \emptyset$

$$F^{\mu\sigma}{}_{,}{}^{\nu} - F^{\mu\nu}{}_{,}{}^{\sigma} - F^{\nu\sigma}{}_{,}{}^{\mu} = \emptyset$$

So $T^{\mu\nu}{}_{,\nu} = \emptyset$

b. For $A_\mu = K_\mu e^{ip_\alpha x^\alpha}$

we have $\partial_\gamma A_\mu = i p_\gamma K_\mu e^{ip_\alpha x^\alpha} = i p_\gamma A_\mu$.

so Lorentz gauge condition: $\partial^\mu A_\mu = i p^\mu K_\mu e^{ip_\alpha x^\alpha} = \emptyset$
tells us that $\boxed{p^\mu K_\mu = \emptyset}$ or $p^\mu A_\mu = \emptyset$

The wave equation is constructed from: $\partial_\gamma \partial_\sigma A_\mu = -p_\gamma p_\sigma K_\mu e^{ip_\alpha x^\alpha}$
then

$$\partial^\sigma \partial_\sigma A_\mu = -p^\sigma p_\sigma K_\mu e^{ip_\alpha x^\alpha} = \emptyset$$

gives $\boxed{p^\sigma p_\sigma = \emptyset}$

c. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = i p_\mu A_\nu - i p_\nu A_\mu$

& from $p_\mu p^\mu = p_\mu A^\mu = \emptyset$, it is clear that $F^{\mu\nu}F_{\mu\nu} = \emptyset$

Then $T^{\mu\nu} = \alpha[F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\emptyset] = -\alpha[(p^\mu A^\sigma - p^\sigma A^\mu)(p^\nu A_\sigma - p_\sigma A^\nu)]$

$$= -\alpha[p^\mu p^\nu A^\sigma A_\sigma]$$

Or $\boxed{T^{\mu\nu} = -\alpha p^\mu p^\nu A^\sigma A_\sigma}$.

Problem 8.2

a. $\vec{A} = A(r)\hat{r}$ magnitude depends on distance from the origin, r, & the vector points radially.

$$\nabla^2\vec{A} = \nabla^2(A(r)\hat{r}) = (\nabla^2 A(r))\hat{r} + A(r)(\nabla^2\hat{r})$$

Now $\nabla^2\hat{r} = \nabla^2\left(\frac{\vec{r}}{r}\right) = (\cancel{\nabla^2\vec{r}})\frac{1}{r} - \frac{\vec{r}}{r^2}\nabla^2 r$

w/ $\nabla^2 r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2) = 2/r \Rightarrow \nabla^2\hat{r} = -2/r^2\,\hat{r}$

So $$\nabla^2\vec{A} = \left[\nabla^2 A(r) + A(r)(-2/r^2)\right]\hat{r}$$

$$= \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A') - 2A/r^2\right]\hat{r} = \phi$$

$$\Downarrow$$

$$\frac{d}{dr}(r^2 A') = 2A$$

Let $A = \alpha r^p$, then $\frac{d}{dr}(r^2 p\alpha r^{p-1}) = 2\alpha r^p$

$$\frac{d}{dr}(p\alpha r^{p+1}) = 2\alpha r^p$$

$$p(p+1)\alpha r^p = 2\alpha r^p$$

So $p(p+1) = 2 \Rightarrow p = 1$ or -2

In general: $\boxed{A(r) = (A_1 r + A_2/r^2)\hat{r}}$

b. We can construct g_{ij} from r_i, the index form of $\vec{r}$, & δ_{ij}

Let $\boxed{g_{ij} = A(r)\,r_i r_j + B(r)\,\delta_{ij}}$

$$ds^2 = dr^i g_{ij} dr^j = A(r)\,r_i r_j dr_i dr_j + B(r)\,dr^i dr^j \delta_{ij}$$

$r_j r_i dr_i dr_j = r^2 dr^2$ & $dr^i dr^j \delta_{ij} = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi$

$$= \boxed{\bar{A}(r)\cdot r^2 dr^2 + \bar{B}(r)\cdot r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

Problem 8.3

For
$$F^{\mu\nu} \doteq \begin{pmatrix} \phi & E_x/c & E_y/c & E_z/c \\ -E_x/c & \phi & B_z & -B_y \\ -E_y/c & -B_z & \phi & B_x \\ -E_z/c & B_y & -B_x & \phi \end{pmatrix} \quad \& \quad F^{\mu\nu}F_{\mu\nu} = -2\,E^2/c^2 + 2B^2$$

$$F_{\mu\nu} \doteq \begin{pmatrix} \phi & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & \phi & B_z & -B_y \\ E_y/c & -B_z & \phi & B_x \\ E_z/c & B_y & -B_x & \phi \end{pmatrix}$$

Spherical symmetry for V & $\vec{A} \longrightarrow V = V(r,t)$ $\vec{A} = A(r,t)\hat{r}$.

Then $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial V}{\partial r}\hat{r} - \dot{A}\hat{r} = (-V' - \dot{A})\hat{r}$ (primes refer to r deriv, dots to t-derivs).

$\vec{B} = \nabla \times \vec{A} = \phi$

&
$$F^{\mu\nu}F_{\mu\nu} = +\frac{2}{c^2}(V' + \dot{A})^2$$

The action is:
$$S = \frac{2\alpha}{c^2}\int dt\,dr\,d\theta\,d\phi\,(r^2 \sin\theta)\,(V' + \dot{A})^2$$
$$= \underbrace{\beta}_{\text{constants}} \int dt\,dr\left[(V' + \dot{A})^2 r^2\right]$$

The Euler-Lagrange equations read:
$$\frac{d}{dr}\left(\frac{\partial \mathcal{L}}{\partial V'}\right) = \phi = \frac{d}{dr}\left[(V' + \dot{A})\cdot 2r^2\right] = \phi$$
&
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{A}} r^2\right) = \phi = \frac{d}{dt}\left[(V' + \dot{A})\cdot 2r^2\right] = \phi.$$

we can choose $A(r,t) = \phi$ (gauge), & we get
$$\frac{d}{dr}\left[2V' r^2\right] = \phi \quad \& \quad \frac{d}{dt}\left[V'\right] = \phi$$

From the first equation, we learn that $V' = \frac{-q(t)}{r^2}$ for arbitrary $q(t)$

so $V = \frac{q(t)}{r}$, & then $\frac{d}{dt}\left(\frac{-q(t)}{r^2}\right) = \phi \Rightarrow q(t) = q_0$

We have: $\boxed{V = q_0/r \quad \& \quad \vec{A} = \phi}$

Problem 8.4

a. For generic: $g^{\mu\nu} \doteq \begin{pmatrix} g^{xx} & g^{xy} \\ g^{xy} & g^{yy} \end{pmatrix}$ functions of (x,y)

we transform to $x' = p(x,y)$, $y' = q(x,y)$, then the metric transforms via:

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}$$

And we require that $g'^{\mu\nu} \doteq f(p,q)\begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}$

so that $g'^{12} = g'^{21} = \phi$

$$g'^{12} = \frac{\partial p}{\partial x}\frac{\partial q}{\partial x} g^{xx} + \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} g^{xy} + \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} g^{xy} + \frac{\partial p}{\partial y}\frac{\partial q}{\partial y} g^{yy} = \phi \qquad (+)$$

$$= \frac{\partial p}{\partial x}\left[\frac{\partial q}{\partial x} g^{xx} + \frac{\partial q}{\partial y} g^{xy}\right] + \frac{\partial p}{\partial y}\left[\frac{\partial q}{\partial x} g^{xy} + \frac{\partial q}{\partial y} g^{yy}\right]$$

Suppose we set:
$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= +\left[\frac{\partial q}{\partial x} g^{xy} + \frac{\partial q}{\partial y} g^{yy}\right] \cdot f(x,y) \\ \frac{\partial p}{\partial y} &= -\left[\frac{\partial q}{\partial x} g^{xx} + \frac{\partial q}{\partial y} g^{xy}\right] f(x,y) \end{aligned}\right\} (*)$$

Then $(+)$ is automatically satisfied.

The second requirement is that $g'^{11} = g'^{22}$,

$$g'^{11} = \frac{\partial p}{\partial x}\frac{\partial p}{\partial x} g^{xx} + \frac{\partial p}{\partial y}\frac{\partial p}{\partial y} g^{yy} + 2\frac{\partial p}{\partial x}\frac{\partial p}{\partial y} g^{xy}$$

$$\& \; g'^{22} = \left(\frac{\partial q}{\partial x}\right)^2 g^{xx} + \left(\frac{\partial q}{\partial y}\right)^2 g^{yy} + 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y} g^{xy}$$

subtracting these, & inputting $(*)$:

$$g^{xx} f^2\left(\frac{\partial q}{\partial x} g^{xy} + \frac{\partial q}{\partial y} g^{yy}\right)^2 + g^{yy}\left(\frac{\partial q}{\partial x} g^{xx} + \frac{\partial q}{\partial y} g^{xy}\right)^2 f^2 - 2f^2 g^{xy}\left(\frac{\partial q}{\partial x} g^{xy} + \frac{\partial q}{\partial y} g^{yy}\right)\left(\frac{\partial q}{\partial x} g^{xx} + \frac{\partial q}{\partial y} g^{xy}\right)$$
$$- \left[\left(\frac{\partial q}{\partial x}\right)^2 g^{xx} + \left(\frac{\partial q}{\partial y}\right)^2 g^{yy} + 2\left(\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\right) g^{xy}\right] = \phi$$

$$= g^{xx} g^{yy}\left[g^{yy}\left(\frac{\partial q}{\partial y}\right)^2 + 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y} g^{xy} + g^{xx}\left(\frac{\partial q}{\partial x}\right)^2 + 2g^{xy}\frac{\partial q}{\partial x}\frac{\partial q}{\partial y} - 2\frac{\partial q}{\partial y}\frac{\partial q}{\partial x} g^{xy}\right] f^2$$
$$- g^{xy} g^{xy}\left[-\left(\frac{\partial q}{\partial x}\right)^2 g^{xx} - \left(\frac{\partial q}{\partial y}\right)^2 g^{yy} + 2\left(\frac{\partial q}{\partial x}\right)^2 g^{xx} + 2\left(\frac{\partial q}{\partial y}\right)\left(\frac{\partial q}{\partial x}\right) g^{xy} + 2\left(\frac{\partial q}{\partial y}\right)^2 g^{yy}\right] f^2$$
$$- \left[\left(\frac{\partial q}{\partial x}\right)^2 g^{xx} + \left(\frac{\partial q}{\partial y}\right)^2 g^{yy} + 2\left(\frac{\partial q}{\partial x}\right)\left(\frac{\partial q}{\partial y}\right) g^{xy}\right] = \phi$$

$$= \underbrace{\left[f^2\left(g^{xx} g^{yy} - g^{xy} g^{xy}\right) - 1\right]}_{f^2 \det g^{\mu\nu} - 1}\left[g^{xx}\left(\frac{\partial q}{\partial x}\right)^2 + 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y} g^{xy} + g^{yy}\left(\frac{\partial q}{\partial y}\right)^2\right] f^2 = \phi$$

Problem 8.4 (continued)

We have:

$$\left(f^2\det(g^{\mu\nu})-1\right)\underbrace{\left[g^{xx}\left(\frac{\partial q}{\partial x}\right)^2+2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}+g^{yy}\left(\frac{\partial q}{\partial y}\right)^2\right]}_{=\det g'^{\mu\nu}\neq\emptyset.}=\emptyset$$

So we must have $f^2\underbrace{\det(g^{\mu\nu})}_{=1/g}-1=\emptyset \Rightarrow f=\sqrt{g}$ w/ $g=\det(g_{\mu\nu})$.

Then (*) reads: $\frac{\partial p}{\partial x}=\sqrt{g}\,\frac{\partial q}{\partial x^\alpha}g^{y\alpha}$ $\quad \frac{\partial p}{\partial y}=-\sqrt{g}\,\frac{\partial q}{\partial x^\alpha}g^{x\alpha}$

& we require cross-derivative equality:

$$\frac{\partial^2 p}{\partial y\partial x}=\left(\sqrt{g}\,\frac{\partial q}{\partial x^\alpha}g^{y\alpha}\right)_{,y}=\frac{\partial^2 p}{\partial x\partial y}=-\left(\sqrt{g}\,\frac{\partial q}{\partial x^\beta}g^{x\beta}\right)_{,x}$$

so $\left(\sqrt{g}\,\frac{\partial q}{\partial x^\alpha}g^{y\alpha}\right)_{,y}+\left(\sqrt{g}\,\frac{\partial q}{\partial x^\beta}g^{x\beta}\right)_{,x}=\left(\sqrt{g}\,\frac{\partial q}{\partial x^\alpha}g^{\gamma\alpha}\right)_{,\gamma}=\emptyset$

by assumption, this is solvable, so, we have satisfied the requirements of conformal flatness:

$$g'^{\mu\nu}=f(p,q)\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

b. Start w/ $g^{\mu\nu}(x,y)=f(x,y)\begin{pmatrix}1&0\\0&1\end{pmatrix}$ w/ $dx^\mu\doteq\begin{pmatrix}dx\\dy\end{pmatrix}$.

the Ricci scalar for this space is (see attached):

$$R=\frac{\nabla f\cdot\nabla f-f\nabla^2 f}{f^3}$$

So for $R=\emptyset$, we need $\nabla f\cdot\nabla f=f\nabla^2 f$

The Riemann tensor has components:

$$R^\alpha{}_{\mu\beta\nu}\sim\nabla f\cdot\nabla f-f\nabla^2 f$$

so if $R=\emptyset$, $R^\alpha{}_{\mu\beta\nu}=\emptyset$ in $D=2$.

In[1]:= `<< /Users/jfrankli/bin/EinsteinVariation.handout.m`

8.4 b

In[2]:=
```
gll = f[x, y] {{1, 0}, {0, 1}};
Xu = {x, y};
```

In[5]:= `GetRicciS[gll, Xu]`

Out[5]=
$$\frac{f^{(0,1)}[x,y]^2 + f^{(1,0)}[x,y]^2 - f[x,y]\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{f[x,y]^3}$$

In[7]:= `Simplify[GetRiemann[gll, Xu]]`

Out[7]=
$$\Bigg\{\Bigg\{\{\{0,0\},\{0,0\}\},\ \Bigg\{\Bigg\{0,\ \frac{f^{(0,1)}[x,y]^2 + f^{(1,0)}[x,y]^2 - f[x,y]\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{2\,f[x,y]^2}\Bigg\},$$
$$\Bigg\{-\frac{f^{(0,1)}[x,y]^2 + f^{(1,0)}[x,y]^2 - f[x,y]\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{2\,f[x,y]^2},\ 0\Bigg\}\Bigg\}\Bigg\},$$
$$\Bigg\{\Bigg\{\Bigg\{0,\ -\frac{f^{(0,1)}[x,y]^2 + f^{(1,0)}[x,y]^2 - f[x,y]\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{2\,f[x,y]^2}\Bigg\},$$
$$\Bigg\{\frac{f^{(0,1)}[x,y]^2 + f^{(1,0)}[x,y]^2 - f[x,y]\left(f^{(0,2)}[x,y] + f^{(2,0)}[x,y]\right)}{2\,f[x,y]^2},\ 0\Bigg\}\Bigg\},\ \{\{0,0\},\{0,0\}\}\Bigg\}\Bigg\}$$

Problem 8.5

From:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \qquad (*)$$

we can hit both sides w/ $g^{\mu\nu}$, that gives:

$$\underbrace{R^{\mu}{}_{\mu}}_{\equiv R} - \frac{1}{2} \underbrace{g^{\mu}{}_{\mu}}_{\delta^{\mu}_{\mu} = D} R = 8\pi \underbrace{T^{\mu}{}_{\mu}}_{\equiv T}$$

so

$$R - \frac{D}{2} R = 8\pi T \quad \rightarrow \quad R = \frac{8\pi T}{1 - D/2}$$

The putting this in for R in $(*)$ gives:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(\frac{8\pi T}{1 - D/2} \right) = 8\pi T_{\mu\nu}$$

or

$$\boxed{R_{\mu\nu} = 8\pi \left(T_{\mu\nu} + \frac{1}{2} \frac{g_{\mu\nu} T}{1 - D/2} \right)}$$

Then for $D = 4$,

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$